MODELLING THE ROOT GROWTH: AN OPTIMAL CONTROL APPROACH TO LINK BIOLOGY AND ROBOTICS

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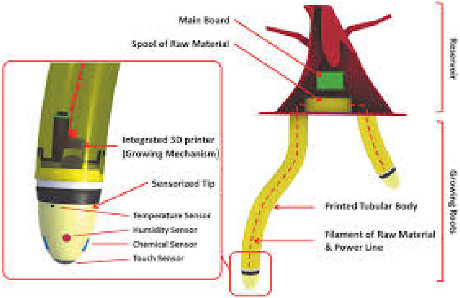
Complex behaviour

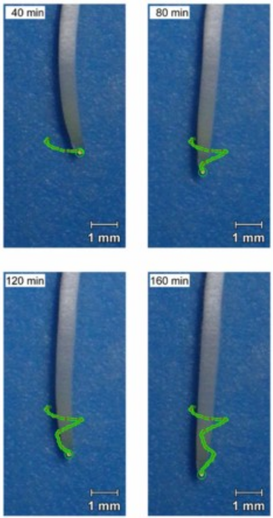
- an activity that requires many **decisions and actions**, in rapid order or simultaneously;
- an activity adopted to **interact** with other organisms and the environment.



CIRCUMNUTATION

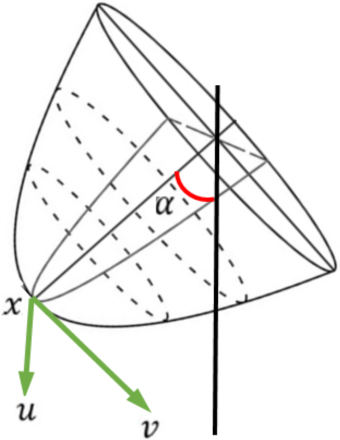
F. Tedone, E. Del Dottore, M. Palladino, B. Mazzolai, P. Marcati,

Optimal control of plant root tip dynamics in soil, submitted



- **Circumnutation** is an elliptical, circular or pendulum like movement;
- Root circumnutation helps **to reduce soil friction** (**Fisher, 1964**);
- **Soil complexity** and different **plant genotypes** make difficult to isolate and study this behaviour (**Bester and Behringe, 2017**);
- To characterise the root circumnutation and **design robots for soil exploration**;
- Modelling of **dynamical** soil-root interactions (**Kolbe, 2017**).

THE MODEL



- x position of the tip;
- v velocity of the tip;
- u control function driving the motion;
- α amplitude of the circumnutation;
- k density from 0 to a fully compressed soil $k^{\max}$;
- $F_s^{\max}$ maximum resistance offered by a compressed soil;
- R function to estimate rearrangement of soil particles.

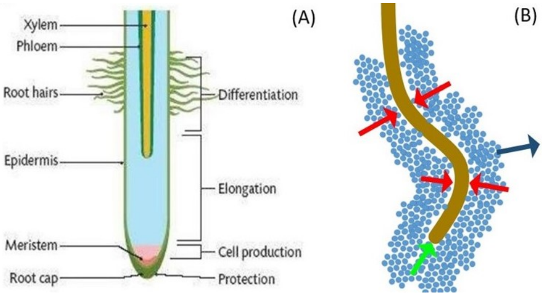
THE MODEL

$$\begin{cases} \dot{x}(t) = v(t), \;\; t \in [t_0, T_f] \\ \dot{v}(t) = u(1 - F_s) - F_d \\ (x(t_0), v(t_0)) = (x_0, v_0) \in \mathbb{R}^6, \\ t_0 \geq 0, \;\; T_f \geq t_0 \end{cases}$$

$$W = \int_{t_0}^{T_f} |\langle u, v\rangle| ds;$$

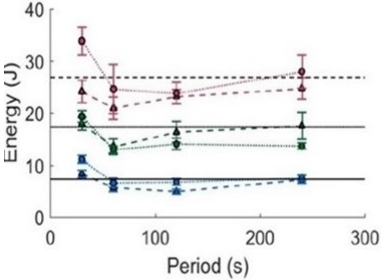
$$F_s(t, v, u) = F_s^{\max} \frac{k}{k^{\max}} R(t, v, u);$$

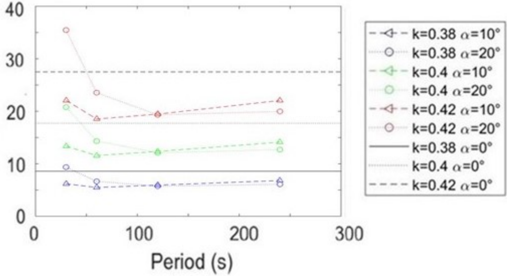
$$F_d(t, v, u) \propto R(t, v, u)v.$$



PARAMETER ESTIMATION

- Constant speed, **straight** descent path;
- Constant speed, **helical circumnutating** descent path;
- Three different densities in **real soil**.

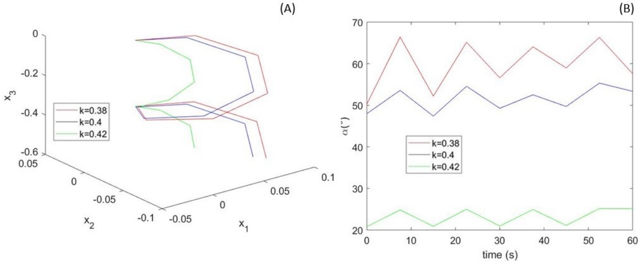




Up to 33% of energy saved by the circumnutation with respect to the straight penetration.

E. Del Dottore et al., *An efficient soil penetration strategy for explorative robots inspired by plant root circumnutation movements*, Bioinspiration & Biomimetics, 2017

THE OPTIMAL CONTROL BASED APPROACH: RESULTS



- Greater oscillations of the optimal trajectory in lower dense soils (**Dexter and Hewitt, 1978, Del Dottore et al., 2016**);
- The amplitude **continuously oscillates** around an optimal value.
- Minimum improvement of 0.16% with respect previous experiments.
- Takes into account **the size** of the tip to support the design of robotic devices;

EFFICIENCY OF MECHANICAL MOTION PATTERNS

F. Tedone, M. Palladino,
Hamilton-Jacobi-Bellman Equation for Control Systems with Friction, submitted

Relaxing the hypothesis...

- Presence of sensors on the robotic tip:

  No regular shape and computation of the interacting surface;

- No homogeneous soil:

  Need to average local soil friction effects;

- Increasing speed for robotic devices:

  The drag force may not depend linearly on the speed

  $$F_d \propto R(t) f(|v|) \frac{v}{|v|};$$

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$$F_d \propto R(t) f(|v|) \frac{v}{|v|};$$

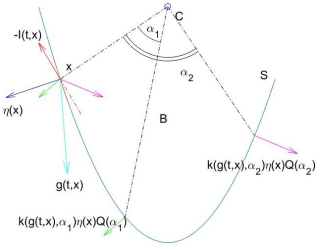
- The dynamics is not Lipschitz $\rightarrow$

Well-posedness?

Existence of minimum solutions?

MATHEMATICAL FRAMEWORK

A vector field $g(t,x)$ is applied to a rigid body $\mathcal{B}$ at x. The friction $I(t,x)$ depends on the **surface of friction** of the rigid body $\mathcal{B}$.



$\eta(x)$ is the normal to $\mathcal{B}$, **transported** along S by $Q(\alpha)$.

We are computing an **average friction**.

$$\dot{x} = g(t,x) - I(t,x);$$

$$I(t,x) = \sum_i k(g(t,x),\alpha_i)\eta(x)\cdot Q(\alpha_i);$$

$$\dot{x} \in g(t,x) - \int_A k(g(t,x),\alpha)\partial_x\varphi(x,\alpha)\mu(d\alpha).$$

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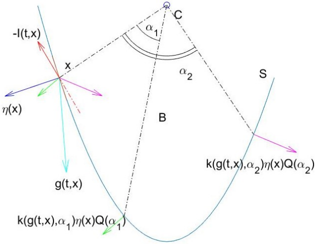

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If both the **vector field** g and the **friction strength** k depend on a **feedback control** u:

$$\dot{x} \in F(t,x,u) = g(t,x,u) - \int_A k(t,x,u,\alpha)\partial_x\varphi(x,\alpha)\mu(d\alpha)$$

MATHEMATICAL FRAMEWORK

Consider the **optimal control** problem:

$$
(P_{t_0,x_0}) \begin{cases} \text{Minimize } W(T, x(T)) \\ \text{over } T > t_0,\ (x,u) \in\ AC([t_0, T]; \mathbb{R}^n) \times \mathcal{U} \\ \dot{x}(t) \in F(t, x(t), u(t)),\ \text{a.e } t \in [t_0, T] \\ u(t) \in U \subseteq \mathbb{R}^m,\ \text{a.e } t \in [t_0, T] \\ x(t_0) = x_0 \in \mathbb{R}^n \\ (T, x(T)) \in \mathrm{Gr}\mathcal{T} \subseteq \mathbb{R}^n \end{cases}
$$

Aim: To characterise the **Value Function** $V(t_0, x_0) = \inf\{(P_{t_0,x_0})\}$.

PROPERTIES OF THE DYNAMICS

The dynamics represents a new class of

- **upper semi-continuous** controlled differential inclusions;
- **not** Lipschitz;
- with a possible **moving** target.

Proposition

$$\bar{F}(t,x) = \bigcup_{u} F(t,x,u)$$

The **dissipative** structure of the system allows to prove that $\bar{F}$ is non-empty, compact and upper semi-continuous. $\bar{F}$ is Lipschitz continuous w.r.t. t and One Sided Lipschitz [a] (OSL) w.r.t. x, uniformly w.r.t. t

[a] T. Donchev, V. Rios, P. Wolenski, *"Strong invariance and one-sided Lipschitz multifunctions"*, Nonlinear Anal.

Existence and uniqueness for any control and initial condition.

PROPERTIES OF THE OPTIMAL CONTROL PROBLEM

- Relaxed **growth condition** for the cost function $W \rightarrow$ Existence of minimisers.

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Define the **reachable** set

$$\mathcal{R}(T;t,x) := \left\{x(T) : \ \dot{x}(s) \in \bar{F}(s,x(s)), \ s \in [t,T], \ x(t) = x\right\}$$

and the set of **admissible trajectories** with initial condition $x(t) = x$:

$$\mathcal{A}(t,x) := \{(T, x(T)) \in \mathrm{Gr}\mathcal{T} : \ x(T) \in \mathcal{R}(T;t,x), \ T \geq t\} .$$

(GC) Fix any $(t,x) \in \mathbb{R}^{1+n}$. For every $(T_k, x_k) \in \mathcal{A}(t,x)$ such that $T_k \to +\infty$, one has that $W(T_k, x_k) \to +\infty$.

GROWTH CONDITIONS

If one replaces **(GC)** with the following condition:

LGC For any $K \subset \mathbb{R}^{1+n}$ compact, there exists $\gamma > 0$ such that

$$W(t', x') \geq W(t, x) + \gamma(t' - t),$$

whenever $(t, x) \in K$, $(t', x') \in \mathcal{A}(t, x)$.

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Then the related optimal solution stops **as soon as** it reaches the target.

(GC) does not imply optimal trajectory stops when it reaches the target.

INWARD POINTING CONDITION

(IPC) *For any compact* $G \subseteq \mathbb{R}^{1+n}$ *there exists* $\rho > 0$ *such that, for all* $(t, x) \in \partial \mathrm{Gr}\mathcal{T} \cap G$

$$\min_{\xi \in \bar{F}(t,x)} \left\{ l^0 + \langle l, \xi \rangle \right\} \leq -\rho \quad \textit{for all} \quad (l^0, l) \in N_{\mathrm{Gr}\mathcal{T}}(t, x).$$

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$$\mathcal{D} = \{(t, x) \in \mathbb{R}^{1+n} : \mathcal{A}(t, x) \neq \emptyset\}$$

Inward Pointing Condition ensures:

$\mathcal{D}$ is open;

the value function V is locally Lipschitz in $\mathcal{D}$.

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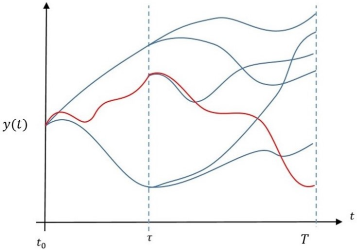
Small Time Controllability of the target.

DYNAMICAL PROGRAMMING PRINCIPLE

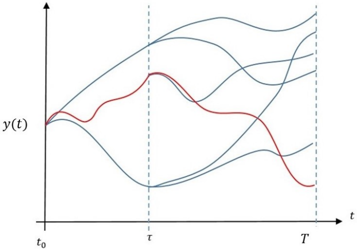


Bellman's principle of Optimality
The second part of an optimal trajectory is optimal.

Dynamic programming principle links the optimal control $\bar{u}$ to the value function V:

- $V(t_0, x_0) \leq V(t, x(t))$ for all $t \in [t_0, T]$, $x(t)$ trajectory for some control $u(t)$;
- $V(t_0, x_0) = V(t, \bar{x}(t))$ for all $t \in [t_0, \bar{T}]$ iff $\bar{x}(t)$ is optimal in $[t_0, \bar{T}]$ with optimal control $\bar{u}(t)$;
- $V(t, x)$ satisfies an appropriate Hamilton Jacobi Bellman **(HJB)** equation.

INVARIANCE PRINCIPLES



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Introduce the minimised and the maximised Hamiltonians:

$$h(t,x,\eta) = \min_{v \in \bar{F}(t,x)} \langle v, \eta \rangle \qquad H(t,x,\eta) = \max_{v \in \bar{F}(t,x)} \langle v, \eta \rangle$$

Invariance principles allows to write the **(HJB)** equation:

- $\mathrm{epi(V)}$ is weakly invariant iff $h(t,x,\eta) \leq 0$ for all $\eta \in N^P_{\mathrm{epi(V)}}(t,x)$;
- $\mathrm{hypo(V)}$ is strongly invariant iff $H(t,x,\eta) \leq 0$ for all $\eta \in N^P_{\mathrm{hypo(V)}}(t,x)$;
- **(HJB)** equation arises if the previous conditions hold.

INVARIANCE PRINCIPLES

$$h(t,x,\eta) = \min_{v \in \bar{F}(t,x)} \langle v, \eta \rangle$$

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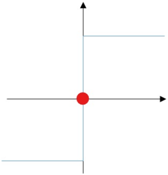

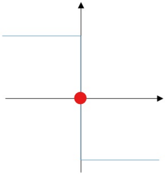

$$\dot{x} \in \begin{cases} 1 & x > 0 \\ [-1,1] & x = 0 \\ -1 & x < 0 \end{cases}$$

$$\dot{x} \in \begin{cases} -1 & x > 0 \\ [-1,1] & x = 0 \\ 1 & x < 0 \end{cases}$$

$\{0\}$ **Weakly Invariant.**

$\{0\}$ **Strongly Invariant.**

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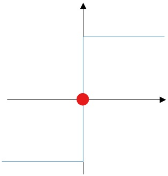

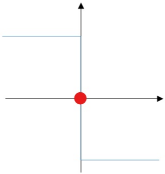

$$\dot{x} \in \begin{cases} 1 & x > 0 \\ [-1,1] & x = 0 \\ -1 & x < 0 \end{cases} \qquad \dot{x} \in \begin{cases} -1 & x > 0 \\ [-1,1] & x = 0 \\ 1 & x < 0 \end{cases}$$

$\{0\}$ **Weakly Invariant.** $\{0\}$ **Strongly Invariant.**

$h(t,0,\eta) \leq 0$, $H(t,0,\eta) \not\leq 0$ for all $\eta \in \mathbb{R}^n$.

HAMILTON-JACOBI-BELLMANN EQUATIONS

Theorem

V is the unique, loc. Lipschitz in $\mathcal{D}$, bounded below viscosity solution of the following problem:
If $(t,x) \in \mathcal{D} \cup \mathrm{Gr}\mathcal{T}^c$,

$$\partial_t V + \liminf_{x' \xrightarrow{-\nabla_x V} x} [h(t,x',\nabla_x V)] = 0.$$

If $(t,x) \in \mathrm{Gr}\mathcal{T}$,

$$\min\left\{ W(t,x) - V(t,x),\ \partial_t V + \liminf_{x' \xrightarrow{-\nabla_x V} x} [h(t,x',\nabla_x V)] \right\} = 0.$$

With the following boundary conditions:

$V(t,x) = +\infty$ for all $(t,x) \notin \mathcal{D}$;

$V(t_k,x_k) \to +\infty$ for all $(t_k,x_k) \to \partial\mathcal{D}$;

$V(t_k,x_k) \to \infty$ for all $(t_k,x_k) \in \mathcal{D}$ such that $t_k \to \infty$.

SUMMARY

- **Validation of the circumnutation** as a mechanical reaction of roots to the soil friction;
- Estimation of **soil forces** in dynamical interactions;
- Development of a tool to support engineers in **designing efficient robots for the soil exploration**;
- Characterisation of the value function for a new class of **non-smooth optimal control problems**.

Thank you
for your attention