UNIVERSITÀ DI TRENTO
Dipartimento di Matematica

11th Conference on
DYNAMICAL SYSTEMS APPLIED TO BIOLOGY AND NATURAL SCIENCES
DSABNS 2020
February 4-7, 2020

Qualitative analysis and numerical approximation of an optimal control model for invasive species

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Invasive species in Protected Areas

wild-boars (*Sus scrofa*)

Feral cats (*Felis catus*)

Alta Murgia
National Park
Italy

Victorian Alpine
National Park
Australia

Tree of heaven (*Ailanthus altissima*)

Improving strategies for the control and eradication of invasive species is an important aspect of nature conservation.

LIFE+ Alta Murgia
(2013-2019).
Control and eradication of *Ailanthus altissima* in the Alta Murgia National Park.

ECOPOTENTIAL H2020 Project
(2015-2019)
Improving Future Ecosystem Benefits Through Earth Observations.

Innonetwork (2019-2020)
Monitoraggio, Allerta e Prevenzione dello stato di Conservazione di Habitat ed ECOsistemi in aree interne e costiere protette e da proteggere.

Piano di gestione triennale del cinghiale nel Parco Nazionale dell'Alta Murgia- Delibera n. 21/2012 del 18/12/2012.

Optimal spatiotemporal control of *Ailanthus altissima* in the Alta Murgia National Park

- Model formulation and well-posedness results
- Numerical approximation
- Simulation and results with Earth Observation data from Remote Sensing
- Sensitivity analysis and validation

Received: 12 February 2018 | Accepted: 9 July 2018
DOI: 10.1111/nrm.12190
WILEY Natural Resource Modeling

Optimal spatiotemporal effort allocation for invasive species removal incorporating a removal handling time and budget

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Baker C.M., Blonda, P., Casella, F., Diele, F., Marangi, C., Martiradonna, A., Montomoli, F. , Pepper, N., Tarantino, C., *Using remote sensing data and expert knowledge within an optimal spatiotemporal model for invasive plant management: the case of* Ailanthus altissima *(Mill.) Swingle in the alta MurgiaNational Park.* In prepatation.

Spatiotemporal optimal control with budget constraint

State PDE

$$\frac{\partial n}{\partial t}(\boldsymbol{x},t) = D\Delta n(\boldsymbol{x},t) + r\, n(\boldsymbol{x},t)\left(\rho(\boldsymbol{x}) - \frac{n(\boldsymbol{x},t)}{k}\right) - \frac{\mu\, n(\boldsymbol{x},t)\, E(\boldsymbol{x},t)}{1 + h\,\mu\, n(\boldsymbol{x},t)}$$

$$n(\boldsymbol{x},t) = n_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \qquad \nabla n \cdot \hat{\boldsymbol{n}} = 0\,, \qquad on\ \partial\Omega \times [0,T]$$

Objective functional

$$J(E) = \int_{\Omega} e^{-\delta T}\, v(\boldsymbol{x})\, n(\boldsymbol{x},T)\, d\boldsymbol{x} + \int_{\Omega\times[0,T]} e^{-\delta t}\,(\,E^2 + \omega\, n)\, d\boldsymbol{x}\, dt + \beta \int_{\Omega\times[0,T]} e^{-\delta t}\left(\frac{E}{B}\right)^3 d\boldsymbol{x}\, dt$$

Control set

$$U = \{E \in L^{\infty}\,(\Omega \times [0,T]) \colon\ 0 \le E \le B\}$$

Well-posedness of the state equation

Assume that $n_0 \in L^\infty(\Omega)$ and $n_0(x) \geq 0$ over Ω.

Then there exists a unique non-negative classical solution of the state PDE.

Existence of the optimal control

There exists an optimal control $E^* \in U$ minimizing the objective functional, i.e.

$$J(E^*) = \min_{E \in U} J(E).$$

Characterization of the optimal control

Assume that E^* is an optimal control and n^*=$n^*(E^*)$ represents its corresponding state. Then there exists a unique solution $\lambda^* \in L^2(0,T;H^1(\Omega))$ for the adjoint equation

$$\frac{\partial\lambda}{\partial t} = -D\Delta\lambda + \delta\lambda - r\rho\lambda + \frac{2rn^*\lambda}{k} + \frac{\mu E^*\lambda}{(1+h\,\mu\, n^*)^2} - \omega$$

$$\lambda(\boldsymbol{x},T) = v(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \quad \nabla\lambda \cdot \hat{\boldsymbol{n}} = 0\,, \quad on\ \partial\Omega \times [0,T]$$

Moreover E^* is characterized by $E^* = \min\left\{\frac{B^3}{3\beta}\left(\sqrt{1 + \frac{3\,\beta\,\mu\, n^*\,\lambda^*}{B^3(1+h\,\mu n^*)}} - 1\right), \mathrm{B}\right\}$

Uniqueness of the optimal control

There exists a positive threshold T_0, depending on the model parameters, such that there is a unique solution (E^*, n^*, λ^*) under the assumption that $0 < T \leq T_0$.

Numerical approximation of the state-adjoint optimality system

$$\frac{\partial n}{\partial t} = D\Delta n + r\,n\left(\rho - \frac{n}{k}\right) - \frac{\mu\,n\,E}{1+h\,\mu\,n}, \qquad n(\boldsymbol{x},0) = n_0(\boldsymbol{x}) \text{ in } \Omega, \qquad \nabla n \cdot \hat{n} = 0 \text{ on } \partial\Omega\text{x}[0,T]$$

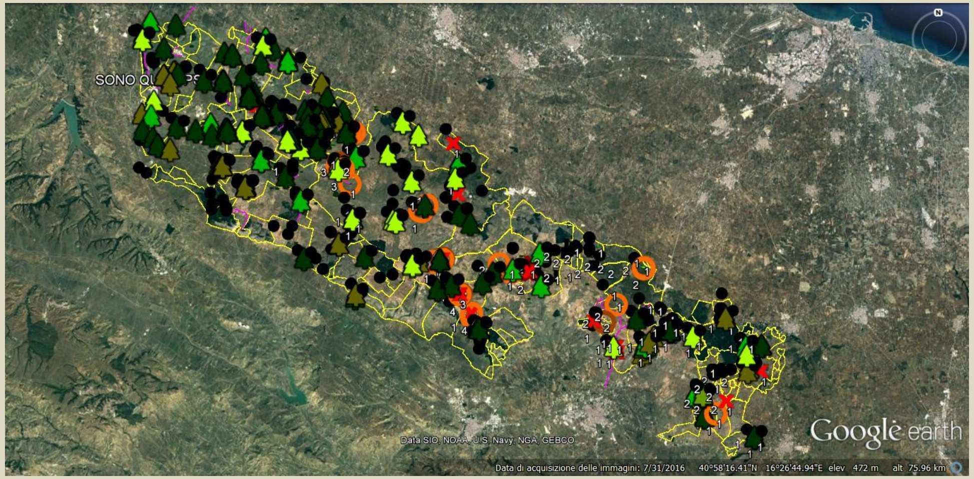
$$\frac{\partial \lambda}{\partial t} = -D\Delta\lambda + \delta\,\lambda - r\,\rho\,\lambda + +2rn\boldsymbol{\lambda} + \frac{\mu\,E\,\lambda}{(1+h\,\mu\,n)^2} - \omega, \qquad \lambda(\boldsymbol{x},T) = v(\boldsymbol{x}) \text{ in } \Omega, \qquad \nabla\lambda \cdot \hat{n} = 0 \text{ on } \partial\Omega\text{x}[0,T]$$

$$E = \min\left\{\frac{B^3}{3\beta}\left(\sqrt{1 + \frac{3\,\beta\,\mu\,n\,\lambda}{B^3(1+h\,\mu\,n)}} - 1\right), \text{B}\right\}$$

- Semi-discretization in the space variable by linear **Finite Elements** or **Finite Differences**
- **Forward-Backward** iterative procedure
- Integration in time by:
  - **Splitting and composition** method for the diffusive and the reaction term
  - **Symplectic-exponential Lawson** method for integrating the reaction term

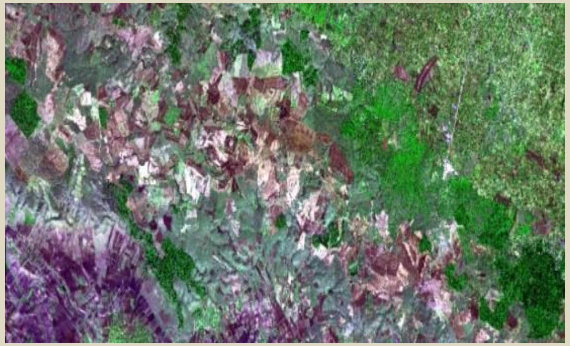
Earth Observation

In-situ data



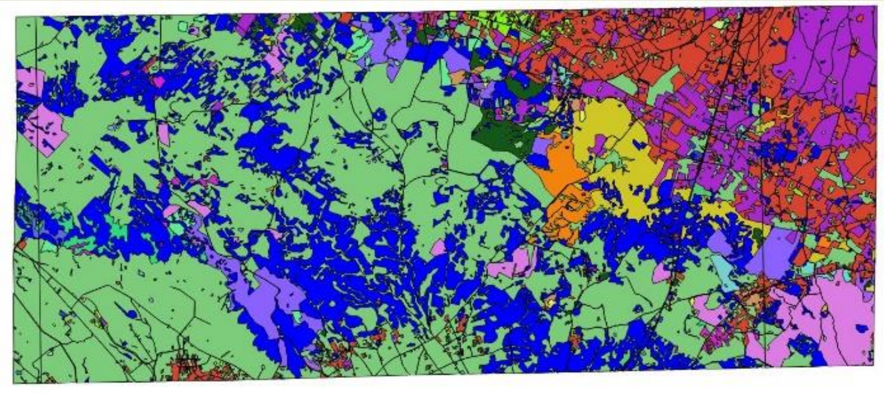
Casella, F., and M. Vurro, **Woody weed mapping in non-crop areas: the case of Ailanthus altissima (tree of Heaven) in Apulia region**. *Giornate Fitopatologiche 2012, Milano Marittima (RA), 13-16 marzo 2012.* (2012): 695-704.

Multi-temporal images from Very High Resolution WorldView-2 satellite (2 meters)

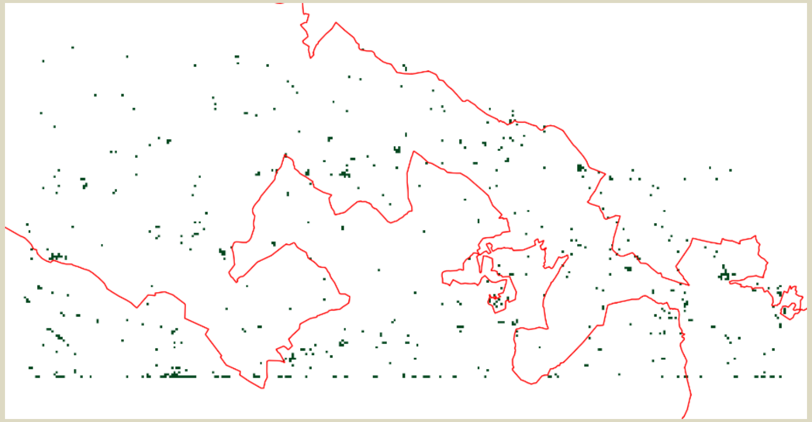


Tarantino, C., Casella, F., Adamo, M., Lucas, R., Beierkuhnlein, C., & Blonda, P. (2019). **Ailanthus altissima mapping from multi-temporal very high resolution satellite images**. ISPRS journal of photogrammetry and remote sensing, 147, 90-103.

Land cover map - Regione Puglia Repository



Presence of Ailanthus



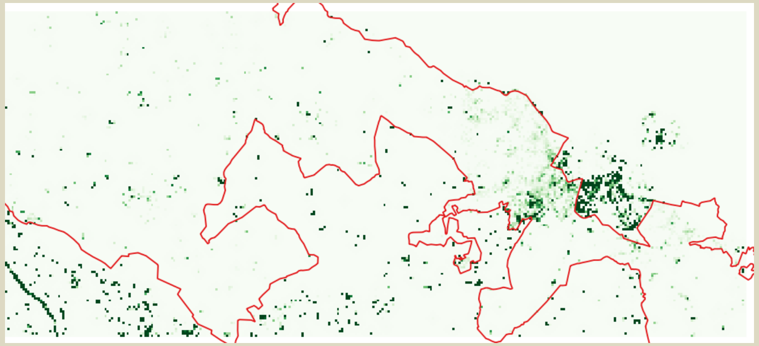
Analyzed area in blue line
(~500 km^2)

Habitat suitability index

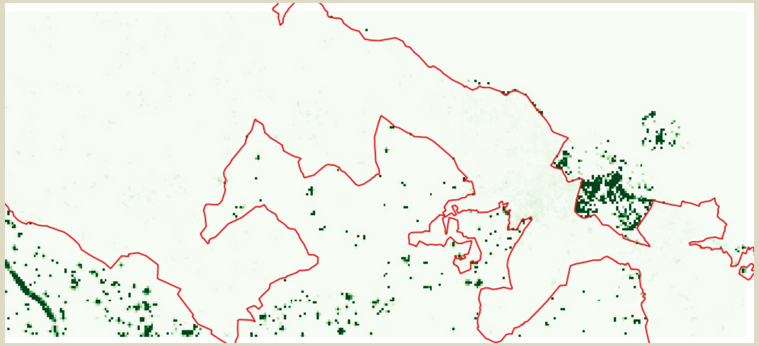
42 land classes (LCCS)	$\rho(x)$
Simple non-irrigated arable land	1
Natural pastures, grassland, uncultivated	0.81
Orchards and small fruit farms	0.21
Olive groves	0.24
Coniferous forest	0.09
…	…
Airports and heliports	0

Simuation. 5 years eradication programme (2014-2019)

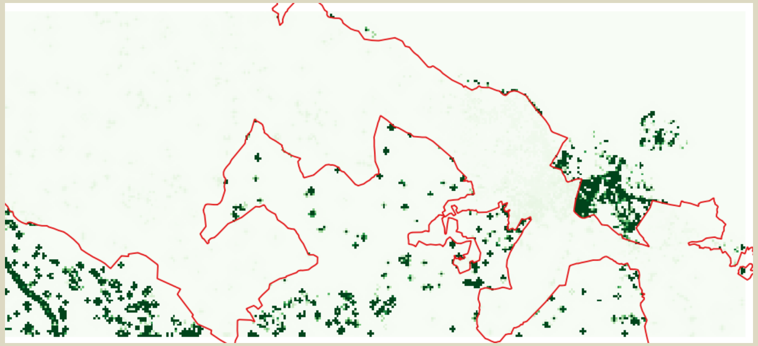
Density 2015



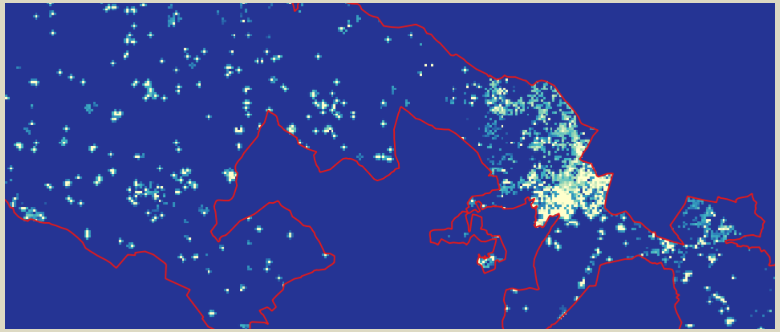
Density 2017



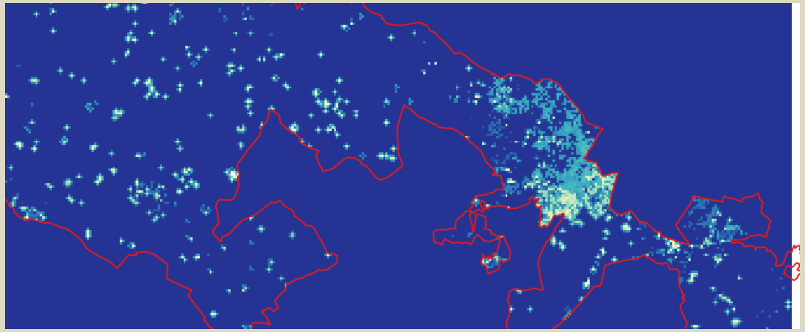
Density 2019



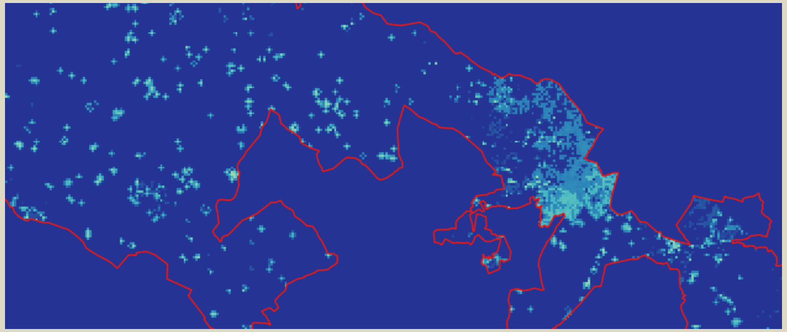
Effort 2015



Effort 2017



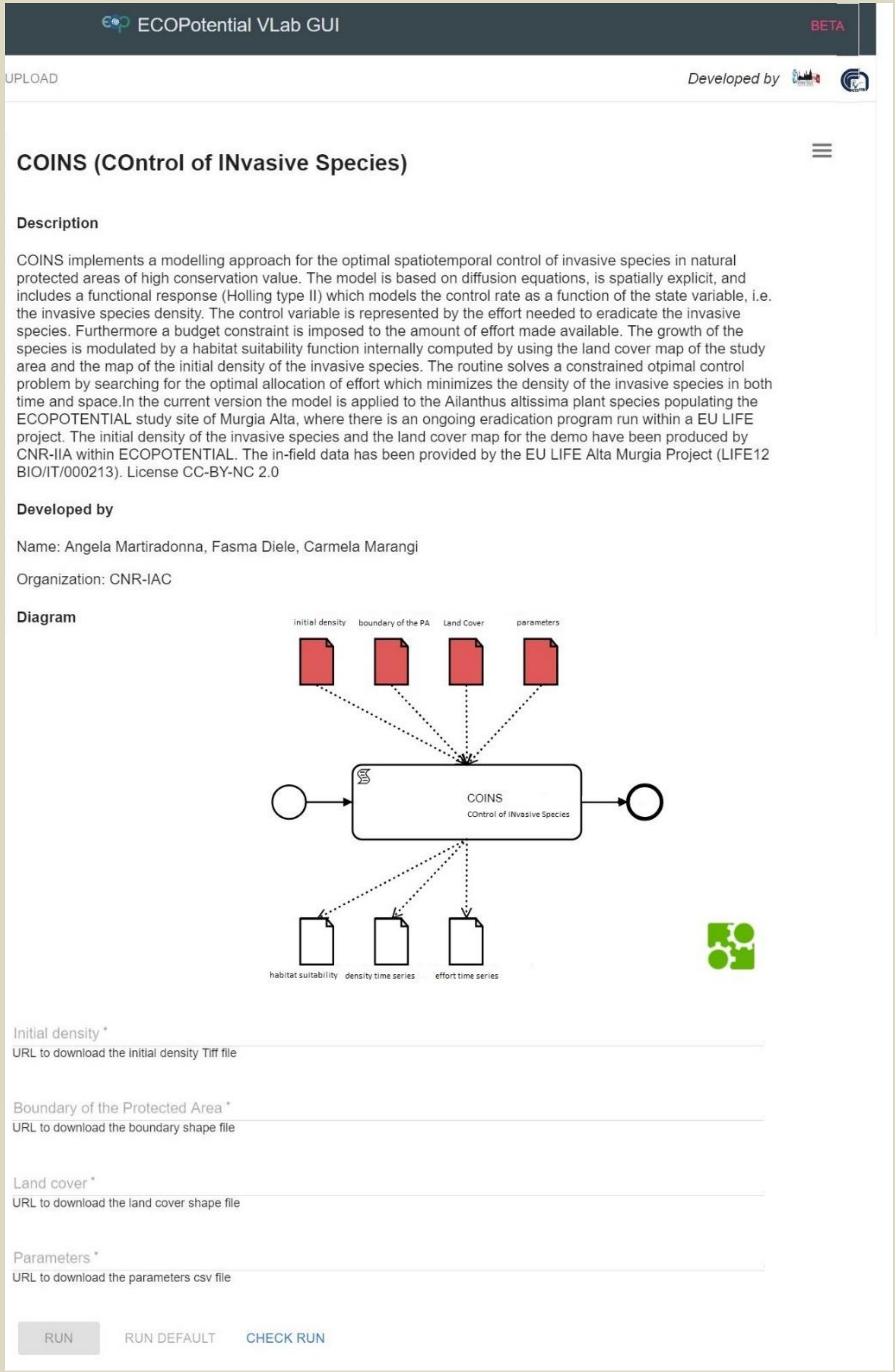
Effort 2019



The routine COINS.R (COntrol of INvasive Species) has been ported on the

ECOPOTENTIAL
Virtual Laboratory Platform (VLab)

and on the GitHub repository
https://github.com/CnrIacBaGit/COINSvlabrepo



Work in progress...

Consistency of the habitat suitability with the species distribution
Boyce index (Hirzel et al., 2006)

Uncertainty quantification with Arbitrary Polynomial Chaos model
(Pepper, Gerardo-Giorda, Montomoli, 2019)

Validation of the reaction-diffusion model

Advection term to model dispersal of seeds by wind

The temporal dynamics

State equation

$$\frac{dn}{dt} = r\,n\left(1 - \frac{n}{k}\right) - \frac{\mu\,n\,E}{1 + h\,\mu\,n}, \quad t \in [0, T]$$

$$n(0) = n_0,$$

Objective functional to be minimized

$$J(E) = \nu\, e^{-\delta\, T} n(T) + \int_0^T e^{-\delta t} \left(E^2 + c \left(\frac{E}{B} \right)^3 \right) dt$$

Control set $\mathcal{U} = \{ E \in L^\infty(0, T) : 0 \le E \le B \}$

Martiradonna A., Diele, F., and Marangi, C. (in press). *Optimal control of invasive species with budget constraint: qualitative analysis and numerical approximation*. In Current Trends in Dynamical Systems in Biology and Natural Sciences. SEMA SIMAI Springer Series.

Current Hamiltonian

$$H(n, E, \lambda) = E^2 + c\left(\frac{E}{B}\right)^3 + \lambda\left(r\,n\left(1 - \frac{n}{k}\right) - \frac{\mu\,n\,E}{1 + h\,\mu\,n}\right)$$

State-(current)costate optimality system

$$\frac{dn}{dt} = \frac{\partial H}{\partial \lambda} = r\,n\left(1 - \frac{n}{k}\right) - \frac{\mu\,n\,E}{1 + h\,\mu\,n}, \quad n(0) = n_0$$

$$\frac{d\lambda}{dt} = \delta\,\lambda - \frac{\partial H}{\partial n} = \delta\,\lambda - r\,\lambda + \frac{2\,r}{k}\,n\,\lambda + \frac{\mu\,E\,\lambda}{(1 + h\,\mu\,n)^2}, \; \lambda(T) = \nu$$

Optimality condition

$$\frac{\partial H}{\partial E} = 0 \Longrightarrow E(t) = \min\{\varphi_\alpha(n(t), \lambda(t)), B\}$$

Forward-backward algorithm

1. Make an initial guess for $E(t)$ over $[0, T]$
2. Solve the state equation forward in time with initial point $n(0) = n_0$
3. Solve the costate equation backward in time with final point $\lambda(T) = \nu$
4. Update E
5. Check convergence:

$$tol\,\|n\| - \|n - n^{old}\| \geq 0$$
$$tol\,\|\lambda\| - \|\lambda - \lambda^{old}\| \geq 0$$

State-(current)costate optimality system

$$\frac{dn}{dt} = \frac{\partial H}{\partial \lambda} = r\,n\left(1 - \frac{n}{k}\right) - \frac{\mu\,n\,E}{1 + h\,\mu\,n}, \quad n(0) = n_0$$

$$\frac{d\lambda}{dt} = \delta\,\lambda - \frac{\partial H}{\partial n} = \delta\,\lambda - r\,\lambda + \frac{2\,r}{k}\,n\,\lambda + \frac{\mu\,E\,\lambda}{(1 + h\,\mu\,n)^2}, \quad \lambda(T) = \nu$$

$$\psi(t) := e^{-\delta\,t}\lambda(t)$$

Nonautonomous Hamiltonian system

$$\frac{dn}{dt} = \frac{\partial \widetilde{H}}{\partial \psi} =: \widetilde{F}(t, n, \psi), \quad \frac{d\psi}{dt} = -\frac{\partial \widetilde{H}}{\partial n} =: \widetilde{G}(t, n, \psi)$$

Symplectic Partitioned Runge-Kutta integrator

$$n_{l+1} = n_l + \Delta t \sum_{i=1}^{s} b_i \, \widetilde{F}(t_l + c_i \, \Delta t, N_i, P_i)$$

$$N_i = n_l + \Delta t \sum_{j=1}^{s} a_{i,j} \, \widetilde{F}(t_l + c_j \, \Delta t, N_j, P_j), \quad i = 1, \ldots, s$$

$$\psi_{l+1} = \psi_l + \Delta t \sum_{i=1}^{s} b_i \, \widetilde{G}(t_l + c_i \, \Delta t, N_i, P_i)$$

$$P_i = \psi_l + \Delta t \sum_{j=1}^{s} \widehat{a}_{i,j} \, \widetilde{G}(t_l + c_j \, \Delta t, N_j, P_j), \quad i = 1, \ldots, s$$

In terms of the current costate $\lambda(t) = e^{\delta t}\,\psi(t)$:

$$n_{l+1} = n_l + \Delta t \sum_{i=1}^{s} b_i\, F(t_l + c_i\,\Delta t, N_i, L_i)$$

$$N_i = n_l + \Delta t \sum_{j=1}^{s} a_{i,j}\, F(t_l + c_j\,\Delta t, N_j, L_j), \quad i = 1, \dots, s$$

$$\lambda_{l+1} = e^{\delta\,\Delta t}\lambda_l - \Delta t \sum_{i=1}^{s} b_i\, e^{\delta\,\Delta t(1-c_i)} G(t_l + c_i\,\Delta t, N_i, L_i)$$

$$L_i = e^{\delta\, c_i\,\Delta t}\lambda_l - \Delta t \sum_{j=1}^{s} \widehat{a}_{ij}\, e^{\delta\,\Delta t(c_i-c_j)} G(t_l + c_j\,\Delta t, N_j, L_j), \quad i = 1, \dots, s$$

where $F(t, n, \lambda) = \dfrac{\partial H}{\partial \lambda}$ and $G(t, n, \lambda) = \dfrac{\partial H}{\partial n}$.

The scheme on λ tuns out to be a Lawson exponential integrator for the dynamics $\dfrac{d\lambda}{dt} = \delta\,\lambda - G(t, n, \lambda)$.

F. Diele, C. Marangi, and S. Ragni (2011). *Exponential Lawson integration for nearly Hamiltonian systems arising in optimal control*. Mathematics and Computers in Simulation, 81(5), 1057-1067.

Backward exponential Lawson

Using the trasversality condition $\lambda_M = \lambda(T) = \nu$, and solving for λ backward in time, we get

$$\lambda_l = e^{-\delta\,\Delta t}\lambda_{l+1} + \Delta t \sum_{i=1}^{s} b_i\, e^{-\delta\, c_i\, \Delta t} G(t_l + c_i\,\Delta t, N_i, L_i)$$

$$L_i = e^{\delta\,(c_i-1)\,\Delta t}\lambda_{l+1} + \Delta t \sum_{j=1}^{s} (b_j - \widehat{a}_{ij})\, e^{\delta\,(c_i-c_j)\,\Delta t} G(t_l + c_j\,\Delta t, N_j, L_j), \quad i = 1, \ldots, s$$

for $l = M-1, \ldots, 0$.

First order Forward-Backward symplectic integrator

The Explicit-Implicit Euler pair gives a first order symplectic RK scheme.

When applied in the Forward-Backward procedure it gives the **explicit-explicit** method:

$$n_{l+1} = n_l + \Delta t\, F(t_l, n_l, \lambda_{l+1}),$$

for $l = 0, \ldots, M-1$,

$$\lambda_l = e^{-\delta\,\Delta t}\left(\lambda_{l+1} + \Delta t\, G(t_l, n_l, \lambda_{l+1})\right),$$

for $l = M-1, \ldots, 0$.

Boundary values: n_0, λ_M. Guess: λ_l, $l = 1, \ldots, M-1$.

Back to the PDE model… splitting

Semi-discretization

$$\mathbf{U}'(t) = -DK\mathbf{U}(t) + \mathbf{F}(\mathbf{U}(t), \mathbf{V}(t)), \qquad \mathbf{U}(0) = \mathbf{U}_0$$
$$\mathbf{V}'(t) = DK\mathbf{V}(t) + \delta\mathbf{V}(t) + \mathbf{G}(\mathbf{U}(t), \mathbf{V}(t)), \qquad \mathbf{V}(T) = \mathbf{V}_T$$

$$\mathbf{U}'(t) = \mathbf{F}(\mathbf{U}(t), \mathbf{V}(t)),$$
$$\mathbf{U}'(t) = -DK\,\mathbf{U}(t),$$
$$\mathbf{V}'(t) = DK\,\mathbf{V}(t),$$
$$\mathbf{V}'(t) = \delta\mathbf{V} + \mathbf{G}(\mathbf{U}(t), \mathbf{V}(t)).$$

Composition of:

- first order explicit exponential Lawson symplectic RK scheme for the local reaction dynamics symplectic RK scheme for the local reaction dynamics
- Implicit Euler scheme for the only diffusive term

Forward-backward IMEX scheme

Boundary values: $\mathbf{U}_0$, $\mathbf{V}_l$, $l = 0, \ldots, M-1$. Guess: $\mathbf{V}_l$, $l = 1, \ldots, M-1$.

1. Forward (explicit-implicit) integration

$$\begin{aligned} \mathbf{U}_{l+1} &= \mathbf{U}_l + \Delta t \; F(t_l, \mathbf{U}_l, \mathbf{V}_{l+1}) \\ (I - \Delta t \; D\,K)\,\mathbf{U}_{l+1} &= \mathbf{U}_l \end{aligned}$$

   for $l = 0, \ldots, M-1$;

2. Backward (implicit-explicit) integration

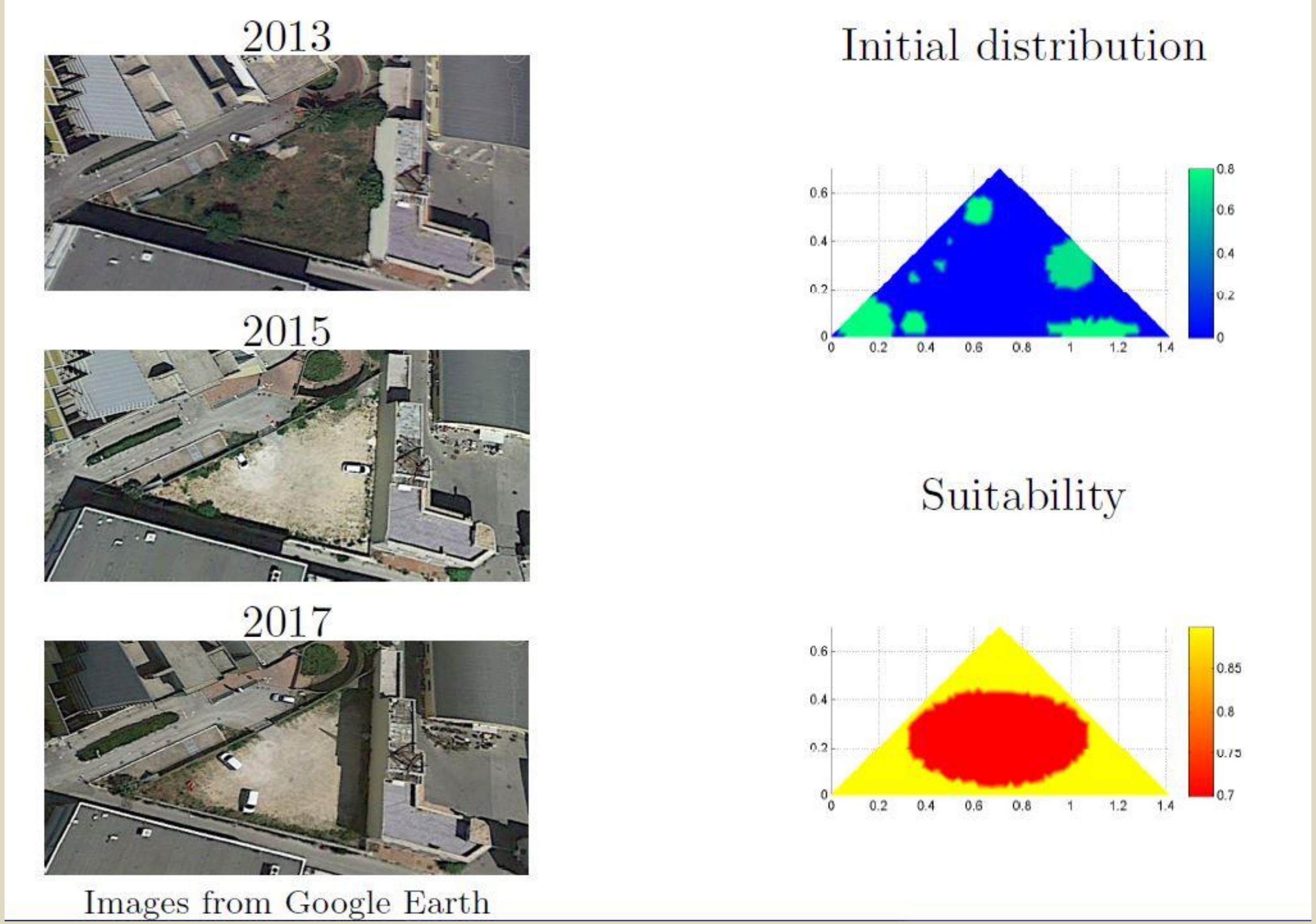
$$\begin{aligned} (I + \Delta t \; D\,K)\,\mathbf{V}_{l+1} &= \mathbf{V}_l \\ \mathbf{V}_l &= e^{-\delta\,\Delta t}\,(\mathbf{V}_{l+1} + \Delta t \; G(t_l, \mathbf{U}_l, \mathbf{V}_{l+1})) \end{aligned}$$

   for $l = M-1, \ldots, 0$;

3. Check for convergence

Finally evaluate $\mathbf{E}_l = \min\{\varphi_\alpha(\mathbf{U}_l, \mathbf{V}_l), B\}$.

Test: invasive plant in a parking area



Space discretization: 512 triangles, $|\Omega| = 1/2$;
Time step: $\Delta t = 1/36$;
Tolerance: $1e - 12$.

T	Iterations
1	18
5	64
10	146
20	265
50	469

Baker, C. M., Diele, F., Marangi, C., Martiradonna, A., & Ragni, S. (2018). Optimal spatiotemporal effort allocation for invasive species removal incorporating a removal handling time and budget. *Natural Resource Modeling*, *31*(4).

Dynamical systems approach


Nonlinear Analysis: Real World Applications 49 (2019) 45–70

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Optimal control of invasive species through a dynamical systems approach

Christopher M. Baker [a,b], Fasma Diele [c,*], Deborah Lacitignola [d], Carmela Marangi [c], Angela Martiradonna [c]

Martiradonna, A., Diele, F., & Marangi, C. (2019). Analysis of state-control optimality system for invasive species management. In Analysis, Probability, Applications, and Computation (pp. 3-13). Birkhäuser, Cham.

Martiradonna A., Diele, F., and Marangi, C. (in press). *Optimal control of invasive species with budget constraint: qualitative analysis and numerical approximation*. In Current Trends in Dynamical Systems in Biology and Natural Sciences. SEMA SIMAI Springer Series.

$$\min_{E \in U} \int_0^T g(t, E)\, dt, \quad U = \{E \in L^1(0, T) : a \leq E \leq b\}$$

subject to

$$\dot{u} = u\, f(u) - u\, (\mu(t)\, E)^q, \quad t \in [0, T]$$

$$u(0) = u_0, \quad u(T) = u_T$$

$u(t)$ the species density, $E(t)$ the control effort,
f continuously differentiable growth function,
g continuously differentiable cost function, convex in E,
$a, b > 0$ fixed constants, $\mu(t) > 0$ scaling function, $q \in \mathbb{Q} \cap \left[\frac{1}{2}, 1\right)$ diminishing returns parameter, $0 < u_T < u_0$ fixed initial and final density.

Case study: wild boars in Alta Murgia National Park

$$f(u) = r \left(\frac{u}{k_0} - 1\right)\left(1 - \frac{u}{k}\right)$$

$$\mu(t) = \mu > 0$$

$$g(t, E) = \mu\, E$$

Baker, Diele, Lacitignola, Marangi, Martiradonna (2019)

Theoretical results:

- Existence of the optimal solution
- First order necessary conditions for optimality
- Uniqueness of the optimal solution

State-adjoint optimality system

$$\dot{u} = \frac{\partial H}{\partial \lambda} = u\, f(u) - u\, (\mu(t)\, E^*)^q\,, \quad u(0) = u_0,\ u(T) = u_T$$

$$\dot{\lambda} = -\frac{\partial H}{\partial u} = -\lambda\, f(u) - \lambda\, u\, f'(u) + \lambda\, (\mu(t)\, E^*)^q$$

$$E^* = \min\{\max\{E, a\}, b\}$$

$$\frac{\partial H}{\partial E}(t, u, E, \lambda) = \frac{\partial g}{\partial E}(t, E) - q\, \mu^q(t)\, \lambda\, u\, E^{q-1} = 0$$

State-control optimality system

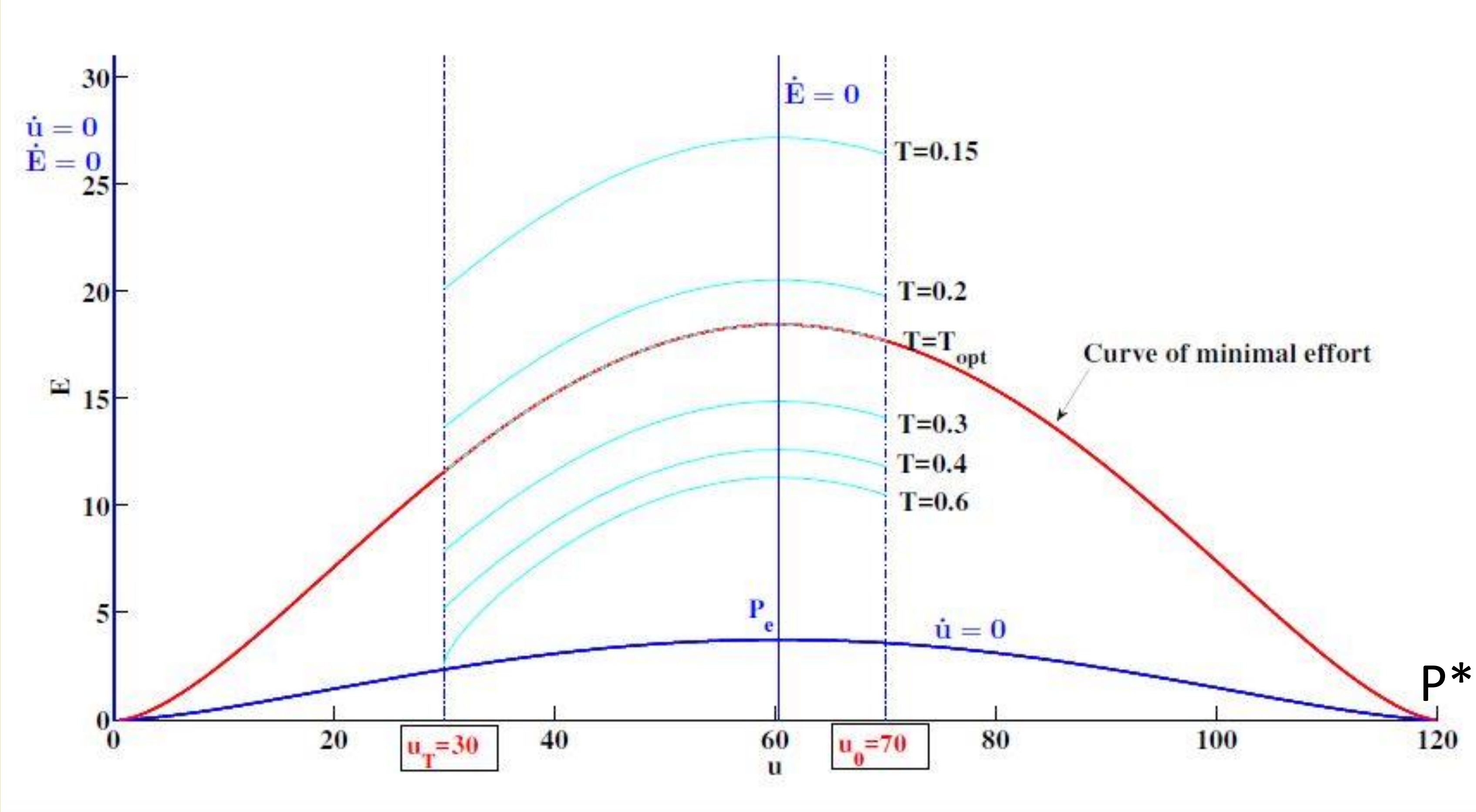
$$\dot{u} = r\, u \left(\frac{u}{k_0} - 1\right)\left(1 - \frac{u}{k}\right) - u\, \mu^q\, E^q, \quad u(0) = u_0,\ u(T) = u_T$$

$$\dot{E} = A\,(2u - k - k_0)\, u\, E$$

$$A = \frac{r}{k\, k_0 (1 - q)} > 0$$

Baker, Diele, Lacitignola, Marangi, Martiradonna (2019)

Qualitative analysis of the state-control dynamics



$$T^* = \frac{1}{A_1} \log \left[\left(\frac{u_T}{u_0} \right)^B \left(\frac{u_0 - k_0}{u_T - k_0} \right)^C \left(\frac{k - u_T}{k - u_0} \right)^D \right]$$

$$\frac{(u^* - k_0)^C}{(u^*)^B (k - u^*)^D} = \frac{(u_T - k_0)^C}{u_T^B (k - u_T)^D} e^{A_1 (T^* - t)},$$

$$E^* = \frac{1}{\mu} \left[\frac{r}{1-q} \left(\frac{u^*}{k_0} - 1 \right) \left(1 - \frac{u^*}{k} \right) \right]^{1/q}$$

Work in progress:

Turnpike property at the saddle point P*

(Ibañez, 2017)

Baker, Diele, Lacitignola, Marangi, Martiradonna (2019)

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Research supported by

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Thank you for the attention