Robustness and fragility of the susceptible-infected-susceptible epidemic chain on networks

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Introduction

- We analyze two modifications of the susceptible-infected-susceptible (SIS) model that preserve its central properties.
- The epidemic thresholds are the same of the original dynamics in heterogeneous (HMF) and quenched (QMF) mean-field theories.
- Simulations yield a dual scenario: the thresholds can be dramatically altered or remain unchanged.

Networks as substrates

- For a pair (i, j), the element of the adjacency matrix is $A_{ij} = 1$ when they are connected and 0 otherwise.
- The degree is given by $k_i = \sum_j A_{ij}$. The moments $\langle k^n \rangle$ are given by the $P_s(k)$.
- A hub has $k \gg \langle k \rangle$. Outliers have degree given by $NP(k) \ll 1$ in an ensemble of networks with degree distribution $P(k)$, $k = k_0, \cdots, k_c$.
- Power-law (PL) distributions $P(k) \sim k^{-\gamma}$, $k \in [k_0, N]$, have $\langle k_{max} \rangle \sim N^{\frac{1}{\gamma-1}}$.
- A structural cutoff is defined here as $k_c = \sqrt{N}$. We can also consider a rigid one as $NP(k_c) = 1$ such that $k_{max} \sim N^{1/\gamma}$. We use $k_0 = 3$.

The SIS epidemic models

In all investigated models, infected vertices are spontaneously healed with rate μ. The infection process, however, is different.

SIS-$\mathcal{T}$ (*threshold*)

- Susceptible vertices become infected with rate λ if they have at least one infected neighbor.

SIS-$\mathcal{A}$ (*all*)

- Infected vertices infect at once all susceptible neighbors with rate λ.

SIS-$\mathcal{S}$ (*standard*)

- Infected vertices independently infect each susceptible neighbor with rate λ.

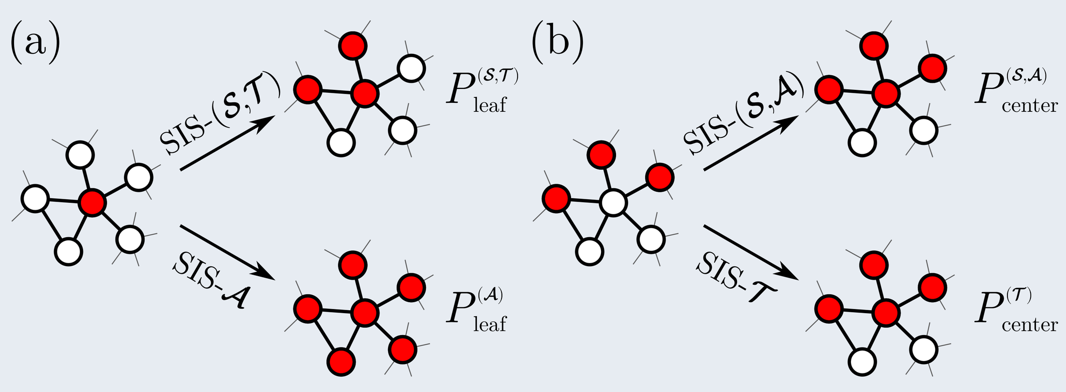

Fig. 1: Some infection processes in the SIS models.

$$P^{(\mathcal{S},\mathcal{T})}_{\text{leaf}}(s) = \binom{k}{s} (\lambda \Delta t)^s (1 - \lambda \Delta t)^{k-s}, \quad (1)$$

$$P^{(\mathcal{A})}_{\text{leaf}}(s) = \lambda \Delta t \delta_{s,k}, \quad (2)$$

$$P^{(\mathcal{S},\mathcal{A})}_{\text{center}}(s) = 1 - (1 - \lambda \Delta t)^s \approx \lambda s \Delta t, \quad (3)$$

$$P^{(\mathcal{T})}_{\text{center}} = \lambda \Delta t. \quad (4)$$

In lattices, all models belong to the directed percolation universality class.

Mean-field analysis

- HMF theory, with $\Theta_k = \sum_{k'} P(k'|k) \rho_{k'}$:

$$\frac{d\rho_k}{dt} = -\mu \rho_k + \lambda (1 - \rho_k) \Psi_k(\Theta_k), \quad (5)$$

where $\Psi_k(\Theta_k) = k\Theta_k$ for SIS-$\mathcal{S}$ and SIS-$\mathcal{A}$, and $\Psi_k(\Theta_k) = 1 - (1 - \Theta_k)^k$ for SIS-$\mathcal{T}$.

- QMF theory:

$$\frac{d\rho_i}{dt} = -\mu \rho_i + \lambda (1 - \rho_i) \Psi_i, \quad (6)$$

where $\Psi_i = \sum_j A_{ij} \rho_j$ for SIS-$\mathcal{S}$ and SIS-$\mathcal{A}$, and $\Psi_i = 1 - \prod_{j|A_{ij}=1} (1 - \rho_j)$ for SIS-$\mathcal{T}$.

- The mean-field thresholds can be obtained by

$$\frac{d\rho_k}{dt} = -\mu \rho_k + \lambda \sum_{k'} C_{kk'} \rho_{k'} \quad (7)$$

and

$$\frac{d\rho_i}{dt} = -\mu \rho_i + \lambda \sum_j A_{ij} \rho_j \quad (8)$$

where $C_{k'k} = kP(k'|k)$.

- *Condition:* largest eigenvalues of the Jacobians $J^{\text{HMF}}_{kk'} = -\mu \delta_{kk'} + \lambda C_{kk'}$ and $J^{\text{QMF}}_{ij} = -\mu \delta_{ij} + \lambda A_{ij}$ are zero.
- Consider $\mu = 1$. For the HMF theory, we have

$$\lambda^{\text{HMF}}_c = \frac{1}{\Upsilon_{\max}}, \quad (9)$$

where $\Upsilon_{\max}$ is the largest eigenvalue of $C_{kk'}$.

- For uncorrelated networks, we obtain

$$\lambda^{\text{HMF}}_c = \frac{\langle k \rangle}{\langle k^2 \rangle}. \quad (10)$$

$$\lambda^{\text{QMF}}_c = \frac{1}{\Lambda_{\max}}, \quad (11)$$

where $\Lambda_{\max}$ is the largest eigenvalue of A_{ij}.

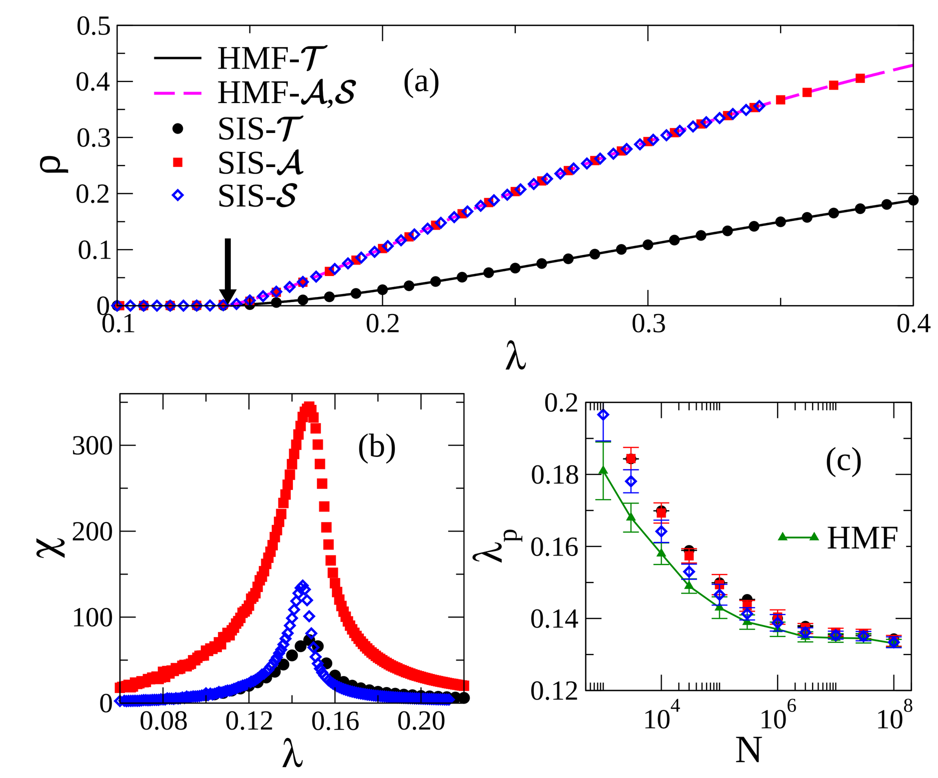

Fig. 2: HMF theory and simulations on annealed networks with $N = 10^5$ and $\gamma = 3.5$. (a) QS density, (b) susceptibility $\chi = N[\langle \rho^2 \rangle - \langle \rho \rangle^2]/\langle \rho \rangle$, and (c) finite-size dependence.

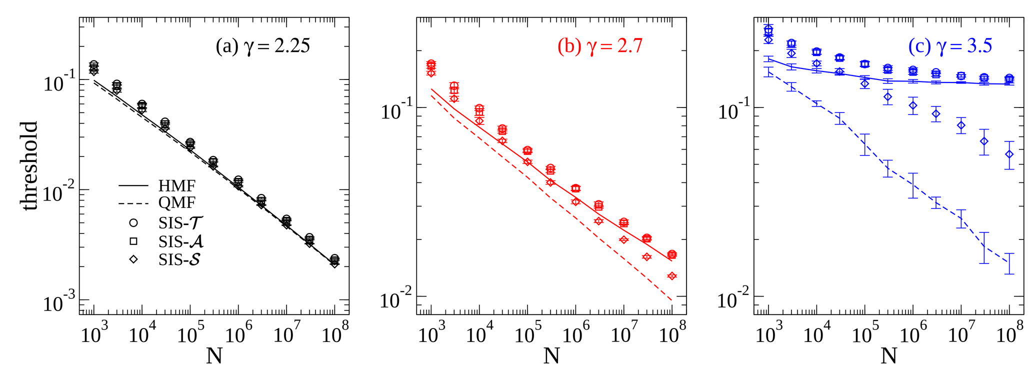

Fig. 3: Thresholds for PL networks with structural cutoff.

Activation mechanisms

- τ_k: recovery time in a star graph of size k.
- τ^{inf}: time that hubs take to mutually transmit the infection to each other.
- If $\tau_k \gg \tau^{\text{inf}}$, the epidemics is triggered by the mutual activation of hubs.
- If $\tau_k \lesssim \tau^{\text{inf}}$, a finite fraction of the network is responsible. For the SIS models, we have:

$$\tau^{\mathcal{S}}_k \approx \frac{2}{\mu} \exp\left(\frac{\lambda^2}{\mu^2} k\right), \quad \tau^{\mathcal{A}}_k \approx \frac{2}{\mu}, \quad \tau^{\mathcal{T}}_k \approx \frac{0.92}{\mu} \ln k$$

Upper bound for uncorrelated networks:

$$\tau^{(\text{inf})}_{kk'} \le \tau_{kk'} = \frac{1}{\lambda} \left[\frac{N \langle k \rangle}{kk'} \right]^{b(\lambda)}, \quad (12)$$

where $b(\lambda) = \ln(1 + \mu/\lambda)/\ln \kappa$ and $\kappa = \langle k^2 \rangle / \langle k \rangle$.

Activation mechanisms for $\gamma > 3$

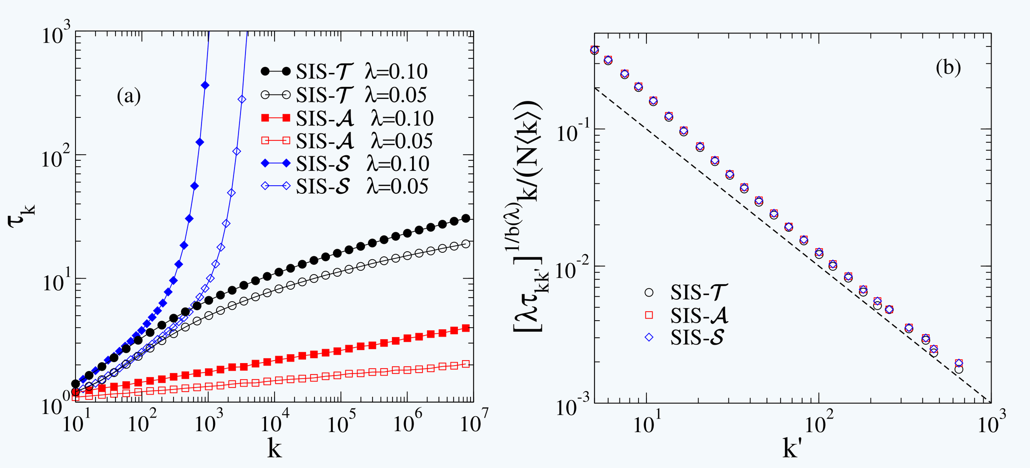

Fig. 4: (a) Activity lifespan on star graphs. 10^3 to 10^5 runs. (b) Mutual reinfection of hubs scaled by Eq. (12). $N = 10^6$, $\gamma = 3.5$, $k = 50$ and $\lambda = 0.05$.

Activation mechanisms for $2 < \gamma < 3$

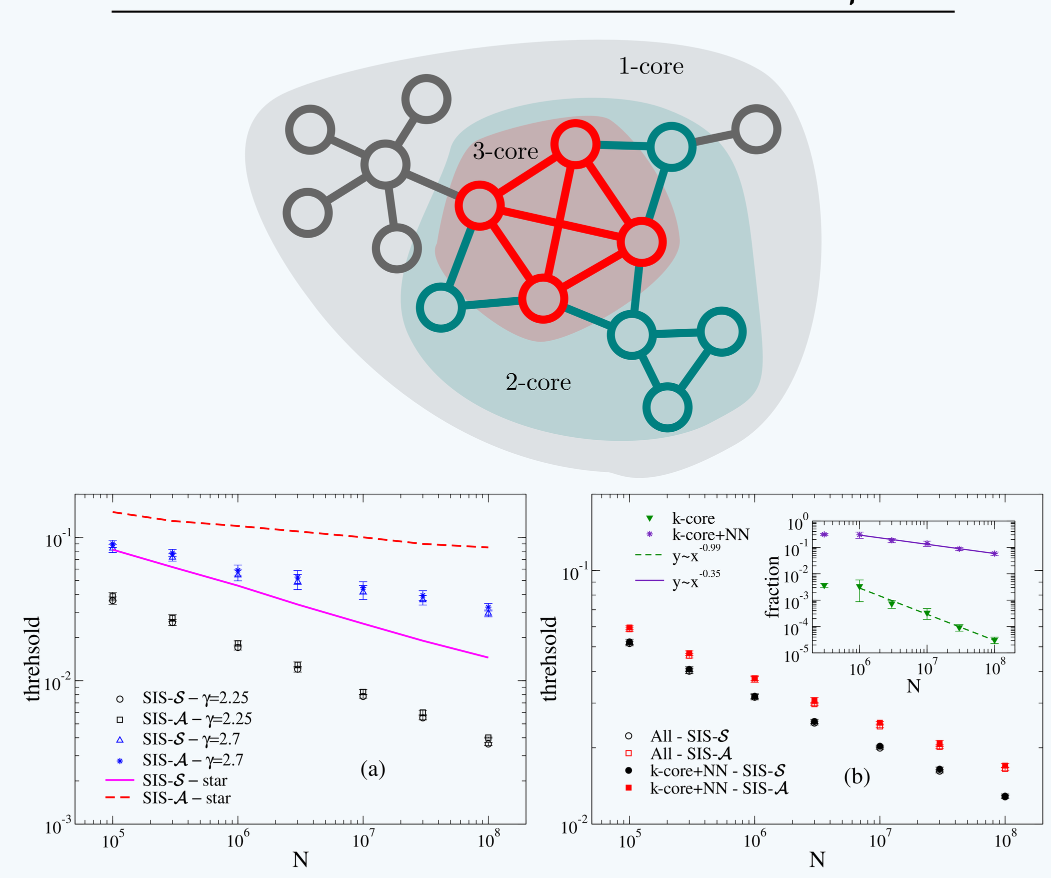

Fig. 5: Epidemic thresholds (a) on the maximum k-core and on star graphs with $k_{\max} \approx \sqrt{N}$; (b) on the max. k-core plus the nearest-neighbors (NN) of a PL network with $\gamma = 2.7$.

Finite-size scaling (FSS)

- We fit the critical QS density and susceptibility as

$$\rho \sim N^{-\nu}, \quad \chi \sim N^{\phi}.$$

- For $\gamma = 2.25$ and 2.7, $k_c = \sqrt{N}$.
- For $\gamma = 3.5$, $k_c \sim N^{1/\gamma}$.

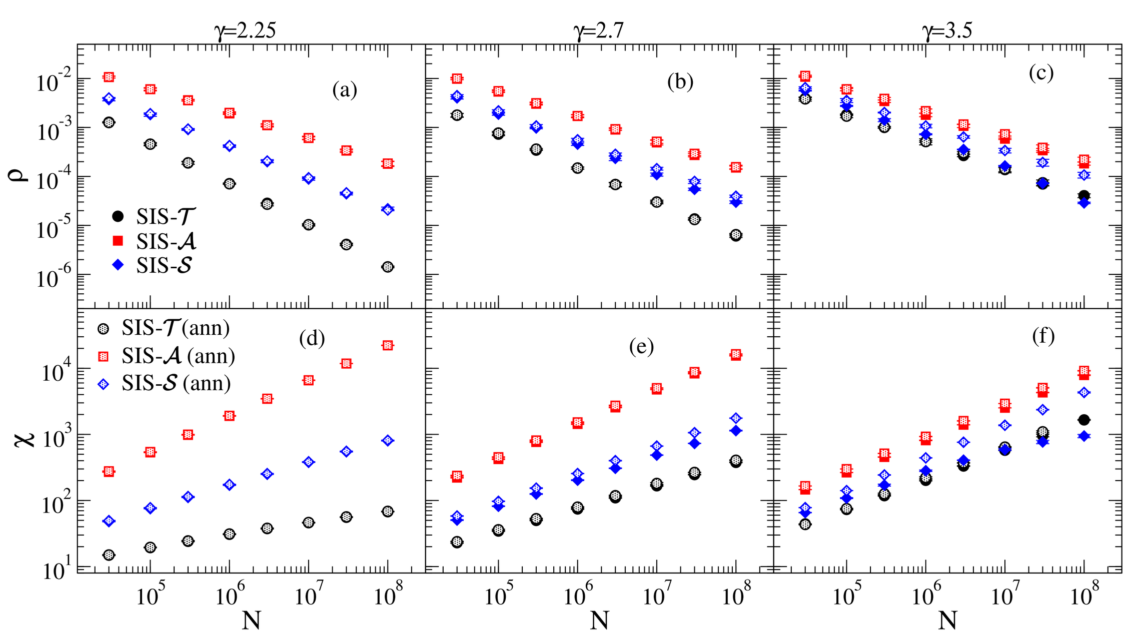

Fig. 6: FSS for the SIS models on PL networks.

Table 1: Critical exponents of the FSS.

Model	$\gamma = 2.25$		$\gamma = 2.7$		$\gamma = 3.5$	
	ν	ν_{ann}	ν	ν_{ann}	ν	ν_{ann}
$\mathcal{T}$	0.845(6)	0.84(2)	0.697(4)	0.692(6)	0.55(1)	0.555(3)
$\mathcal{A}$	0.519(9)	0.517(4)	0.52(1)	0.515(9)	0.499(6)	0.49(3)
$\mathcal{S}$	0.63(2)	0.655(2)	0.60(2)	0.57(1)	–	0.506(7)
	ϕ	ϕ_{ann}	ϕ	ϕ_{ann}	ϕ	ϕ_{ann}
$\mathcal{T}$	0.167(2)	0.169(1)	0.353(1)	0.352(1)	0.458(1)	0.467(3)
$\mathcal{A}$	0.530(2)	0.528(2)	0.514(1)	0.513(1)	0.494(1)	0.497(1)
$\mathcal{S}$	0.329(5)	0.329(4)	0.372(1)	0.421(1)	–	0.496(1)

Conclusions

Table 2: Activation mechanisms for different epidemic models.

Model	$2 < \gamma < 5/2$	$5/2 < \gamma < 3$	$\gamma > 3$
SIS-$\mathcal{S}$	max k-core	hub	hub
SIS-$\mathcal{T}$	max k-core	max k-core	collective
SIS-$\mathcal{A}$	max k-core	max k-core	collective
SIRS	max k-core	max k-core	collective
CP	collective	collective	collective

- For $2 < \gamma < 3$, there is a null threshold irrespective of the existence of locally activated hubs.
- The metastable, localized, and active states of the SIS-$\mathcal{S}$ for $\gamma > 3$ are not universal and may be unrealistic.

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Acknowledgments

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