A GSPT approach to epidemics on homogeneous graphs - DSABNS 2020

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The setting:

- homogeneous connected graph with N nodes (normalize to 1), each with degree n
- three possible states for nodes: S, I, R (densities)
- no birth, no death, no creation/deletion of edges
- infection rate τ (spreading through edges SI)
- recovery rate $\gamma < \tau$
- loss of immunity at rate δ

Visual examples of homogeneous graphs

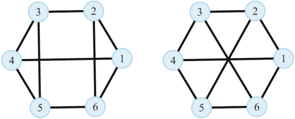


Figure: Stolen from KMS

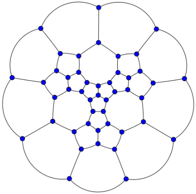


Figure: Stolen from the internet

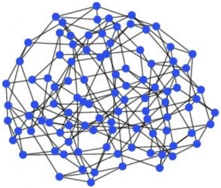


Figure: Idem

Introduction to epidemics on graphs 2/3

ODEs governing the nodes:

$$
\begin{aligned}
[\dot{S}] &= -\tau[SI] + \delta[R] \\
[\dot{I}] &= \tau[SI] - \gamma[I] \\
[\dot{R}] &= \gamma[I] - \delta[R]
\end{aligned}
$$

which involve edge $[SI]$ (only one on which infection can spread).
"Standard" *SIRS* model without demography, except for terms $[SI]$ instead of $[S][I]$.

Introduction to epidemics on graphs 3/3

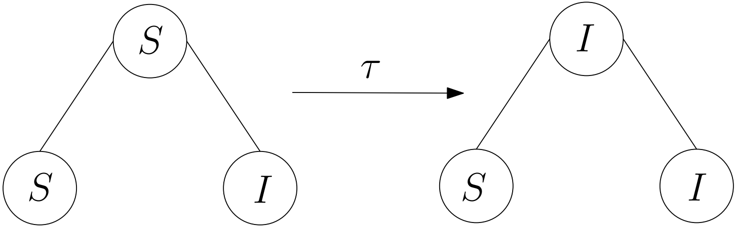


Figure: Example of role of triples

Equation governing $[SI]$:

$$\dot{[SI]} = -(\gamma + \tau)[SI] + \delta[IR] + \tau[SSI] - \tau[ISI]$$

System of ODEs

$$
\begin{aligned}
[\dot{S}] &= -\tau[SI] + \delta[R] \\
[\dot{I}] &= \tau[SI] - \gamma[I] \\
[\dot{R}] &= \gamma[I] - \delta[R] \\
[\dot{SS}] &= \delta[SR] - 2\tau[SSI] \\
[\dot{SI}] &= -(\gamma+\tau)[SI] + \delta[IR] + \tau[SSI] - \tau[ISI] \\
[\dot{SR}] &= \gamma[SI] - \delta[SR] + 2\delta[RR] - \tau[ISR] \\
[\dot{II}] &= \tau[SI] - 2\gamma[II] + \tau[SSI] + \tau[ISI] \\
[\dot{IR}] &= 2\gamma[II] - (\gamma+\delta)[IR] + \tau[ISR] \\
[\dot{RR}] &= \gamma[IR] - 2\delta[RR]
\end{aligned}
$$

Note: sum of nodes is constant ($=1$), sum of edges too ($=\frac{n}{2}$).
We remove $[R]$ and $[RR]$.

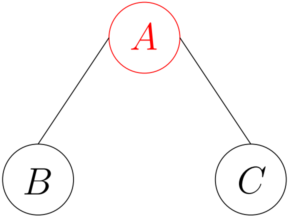
How does $[SSI]$ (or any other triple) evolve in time?
To fully describe dynamics, would need following ODEs:

- 3 equations for possible states of the nodes, involve edges
- 6 (or 9) equations for edges, involve triples
- equations for triples, involve quadruples
- equations for quadruples, involve quintuples
- ...

Need to truncate somewhere: moment closures.

Moment closures 2/2

Choice 1: approximate edges. $[AB] \approx \frac{n}{N}[A][B]$ (up to rescaling τ, obtain model studied in [arXiv])
Choice 2: approximate triples.



$$[BAC] \approx \frac{n-1}{n}\frac{[BA][AC]}{[A]} \tag{1}$$

System of ODEs

$$
\begin{aligned}
\dot{[S]} &= -\tau[SI] + \delta(1 - [S] - [I]) \\
\dot{[I]} &= \tau[SI] - \gamma[I] \\
\dot{[SS]} &= \delta[SR] - 2\tau\frac{n-1}{n}\frac{[SS][SI]}{[S]} \\
\dot{[SI]} &= -(\gamma+\tau)[SI] + \delta[IR] + \frac{n-1}{n}\tau[SI]\left(\frac{[SS]}{[S]} - \frac{[SI]}{[S]}\right) \\
\dot{[SR]} &= \gamma[SI] - \delta[SR] + 2\delta\left(\frac{n}{2} - [SS] - [SI] - [SR] - [II] - [IR]\right) \\
&\quad - \tau\frac{n-1}{n}\frac{[SI][SR]}{[S]} \\
\dot{[II]} &= \tau[SI] - 2\gamma[II] + \frac{n-1}{n}\tau[SI]\left(\frac{[SS]}{[S]} + \frac{[SI]}{[S]}\right) \\
\dot{[IR]} &= 2\gamma[II] - (\gamma+\delta)[IR] + \tau\frac{n-1}{n}\frac{[SI][SR]}{[S]}
\end{aligned}
$$

Fast-slow dynamics 1/5

Want to have loss of immunity slow, infection and recovery fast: replace δ with $\epsilon\delta$, $0 < \epsilon \ll 1$.

$$[\dot{S}] = -\tau[SI] + \epsilon\delta(1 - [S] - [I]), \qquad [\dot{I}] = \tau[SI] - \gamma[I]$$

$$[\dot{SS}] = \epsilon\delta[SR] - 2\tau\frac{n-1}{n}\frac{[SS][SI]}{[S]}$$

$$[\dot{SI}] = -(\gamma+\tau)[SI] + \epsilon\delta[IR] + \frac{n-1}{n}\tau[SI]\left(\frac{[SS]}{[S]} - \frac{[SI]}{[S]}\right)$$

$$[\dot{SR}] = \gamma[SI] - \epsilon\delta[SR] + 2\epsilon\delta\left(\frac{n}{2} - [SS] - [SI] - [SR] - [II] - [IR]\right) - \tau\frac{n-1}{n}\frac{[SI][SR]}{[S]}$$

$$[\dot{II}] = \tau[SI] - 2\gamma[II] + \frac{n-1}{n}\tau[SI]\left(\frac{[SS]}{[S]} + \frac{[SI]}{[S]}\right)$$

$$[\dot{IR}] = 2\gamma[II] - (\gamma + \epsilon\delta)[IR] + \tau\frac{n-1}{n}\frac{[SI][SR]}{[S]}$$

Fast-slow dynamics 2/5

Critical manifold: $[I] = [SI] = [II] = [IR] = 0$.
Slow dynamics on slow manifold $[I] = [SI] = [II] = [IR] = 0$:

$$\begin{aligned}
\dot{[S]} &= \epsilon\delta(1-[S]) \\
\dot{[SS]} &= \epsilon\delta[SR] \\
\dot{[SR]} &= \epsilon n\delta - 3\epsilon\delta[SR] - 2\epsilon\delta[SS]
\end{aligned} \tag{2}$$

Rescale time:

$$\begin{aligned}
[S]' &= \delta(1-[S]) \\
[SS]' &= \delta[SR] \\
[SR]' &= n\delta - 3\delta[SR] - 2\delta[SS]
\end{aligned} \tag{3}$$

In particular,

$$([SS]+[SR])' = 2\delta\left(\frac{n}{2} - ([SS]+[SR])\right) \tag{4}$$

$[S] \to 1$, $[SS]+[SR] \to \frac{n}{2}$

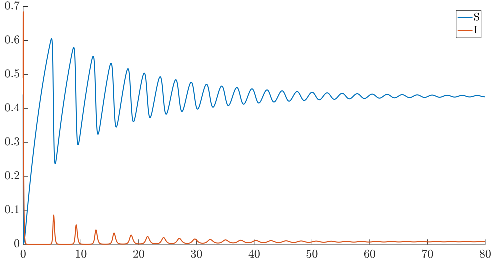
Can be shown: slow manifold exponentially close to critical manifold
Eigenvalues of Jacobian on the critical manifold, for $\epsilon = 0$:

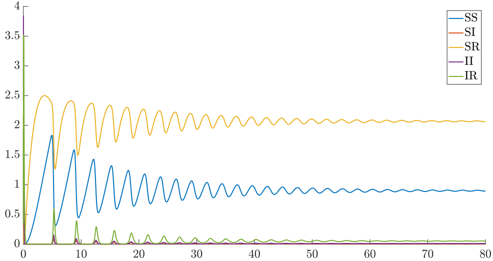
$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$\lambda_4 = \lambda_5 = \frac{\lambda_6}{2} = -\gamma < 0$$

$$\lambda_7 = \frac{(n\tau - \gamma)[SS]}{n[S]} - (\gamma + \tau)$$

Hyperplane on which λ_9 changes sign: $(n\tau - \gamma)[SS] - n(\gamma + \tau)[S] = 0$
$n\tau - \gamma > 0$ since $\tau > \gamma$ and $n > 1$ (if $n = 1$ graph not connected, just couples)
We expect: damped oscillations towards unique equilibrium


S
I
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
70
80


SS
SI
SR
II
IR
4
3.5
3
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80

Idea: rescale $[I]=\epsilon[v]$ when $[I]=\mathcal{O}(\epsilon)$ (as in [arXiv])
$(\implies [SI]=\epsilon[Sv],\ [IR]=\epsilon[vR],\ [II]=\epsilon^2[vv])$

$$[\dot{S}] = -\epsilon\tau[Sv]+\epsilon\delta(1-[S]-\epsilon[v]) \quad \textit{slow} \qquad \epsilon[\dot{v}] = \epsilon\tau[Sv]-\epsilon\gamma[v]$$

$$[\dot{SS}] = \epsilon\delta[SR]-2\epsilon\tau\frac{n-1}{n}\frac{[SS][Sv]}{[S]} \quad \textit{slow} \qquad \uparrow \textit{fast}$$

$$\epsilon[\dot{Sv}] = -\epsilon(\gamma+\tau)[Sv]+\epsilon^2\delta[vR]+\epsilon\frac{n-1}{n}\tau[Sv]\left(\frac{[SS]}{[S]}-\epsilon\frac{[Sv]}{[S]}\right) \quad \textit{fast}$$

$$[\dot{SR}] = \epsilon\gamma[Sv]-\epsilon\delta[SR]+2\epsilon\delta\left(\frac{n}{2}-[SS]-\epsilon[Sv]-[SR]-\epsilon^2[vv]\right.$$
$$\left.-\epsilon[vR]\right)-\tau\epsilon\frac{n-1}{n}\frac{[Sv][SR]}{[S]} \quad \textit{slow}$$

$$\epsilon^2[\dot{vv}] = \epsilon\tau[Sv]-2\epsilon^2\gamma[vv]+\epsilon\frac{n-1}{n}\tau[Sv]\left(\frac{[SS]}{[S]}+\epsilon\frac{[Sv]}{[S]}\right) \quad \textit{superfast}$$

$$\epsilon[\dot{vR}] = 2\epsilon^2\gamma[vv]-\epsilon(\gamma+\epsilon\delta)[vR]+\epsilon\tau\frac{n-1}{n}\frac{[Sv][SR]}{[S]} \quad \textit{fast}$$

References

- C. Kuehn. *Multiple time scale dynamics*, volume 191. Springer, 2015.
- I. Z. Kiss, J. Miller, P. L. Simon, *Mathematics of Epidemics on Networks*, IAM, volume 46. Springer, 2017.
- H. Jardón-Kojakhmetov, C. Kuehn, A. Pugliese, M. Sensi. *A geometric analysis of the SIR, SIRS and SIRWS epidemiological models*, arXiv: `https://arxiv.org/abs/2002.00354`

Thank you for your attention!