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GLOBAL STABILITY ANALYSIS OF BIRHYTHMICITY IN A VAN DER POL TYPE SELF-SUSTAINED OSCILLATOR

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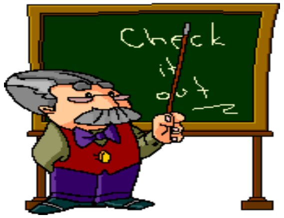
Outline

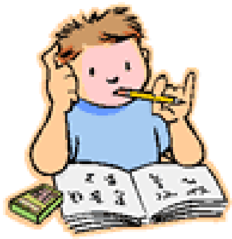


- Birhythmic oscillators
- Global versus local stability
- Basic structure of the van der Pol-type oscillations
- Effective activation energies
- Numerical and analytical estimates
- Residence times and analogy with phase transitions
- Conclusions and Perspectives

This works deals with attractors

Attractors, or cycles, or orbits, or rhythms, are ubiquitous in biological and natural sciences.

Natural systems oscillates because of



- *Circadian, seasonal forcing*
- *Population competition*
- *Chemical instabilities*

Main question: if more than an orbit is possible, how to asses the relative stability ?

Applications of birhytmicity

Birhythmic oscillations, e.g. two stable frequencies as in:

[1] Chemical oscillating reactions

[2] Enzyme-reactions (very indirect)

[3] Cats neural electro-activity

[4] Circadian rhythms (not clear cut evidence)

[5] Superconducting circuits coupled to a resonator (Josephson junctions)

[1] M Alamgirm I. R. Epstein, *Birhythmicity and Compound Oscillation in Coupled Chemical Oscillators: Chlorite-Bromate-Iodide* J. Am. Chem. SOC. **105**, 2500-2502(1983)

System [2] Froelich, H. (1986) Coherence and the action of enzymes. In: Welch, G.R. Editor, *The Fluctuating Enzyme*. New York, Wiley, p. 421; Kaiser, F. (1977). *Cycle model for brain waves* Biol. Cybernetics 27, 155.; Goldbeter, A. (2002) *Computational approaches to cellular rhythms*. Nature 420, 238-245; Goldbeter A. (1996). *Biochemical Oscillations and Cellular Rhythms. The Molecular Bases of Periodic and Chaotic Behaviour* Cambridge University Press; Kar, S., Ray, D.S. *Large fluctuations and nonlinear dynamics of birhythmicity*. Europhys. Lett. 67, 137–143 (2004)

[5] Jewett, M.E., Forger, D.B., Kronauer, R.E. (1999). *Revised Limit Cycle Oscillator Model of Human Circadian Pacemaker*. J. Biol. Rhythms 14, 493-499.

[3] J. Hounsgaard, H. Hultborn, B. Jespersen, and O. Kiehn, J. Physiol. 405, 345 (1988).

[4] Jewett, M.E., Forger, D.B., Kronauer, R.E.: *Revised Limit Cycle Oscillator Model of Human Circadian Pacemaker.* J. Biol. Rhythms 14, 493–499 (1999

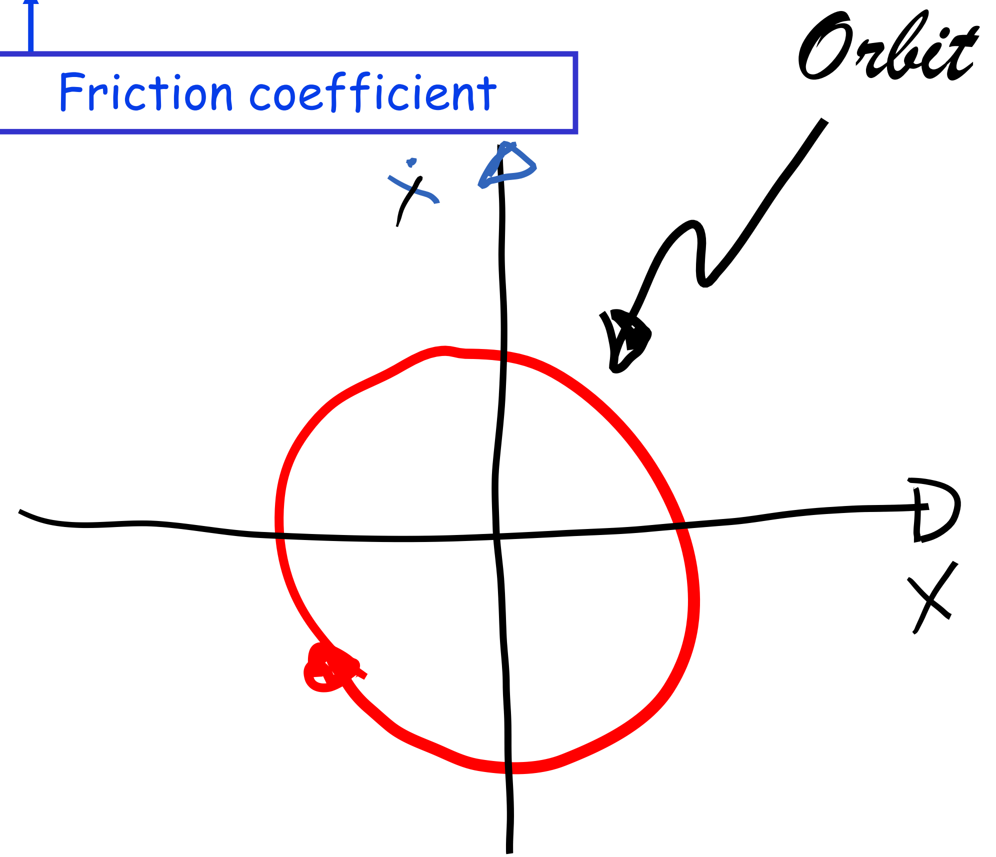
[5] R. Yamapi, G. Filatrella, *Noise effects on a birhythmic Josephson junction coupled to a resonator*, Phys. Rev. E 89, 052905 (2014); O. V. Pountougnigni, R. Yamapi, G. Filatrella, C. Tchawoua, *Noise and disorder effects in a series of birhythmic Josephson junctions coupled to a resonator*, Phys. Rev. E 99, 032220-01-10 (2019)

Van der Pol oscillators:

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + x = 0$$

Friction coefficient

Self oscillations appear, as the coefficient of the friction is positive for small x

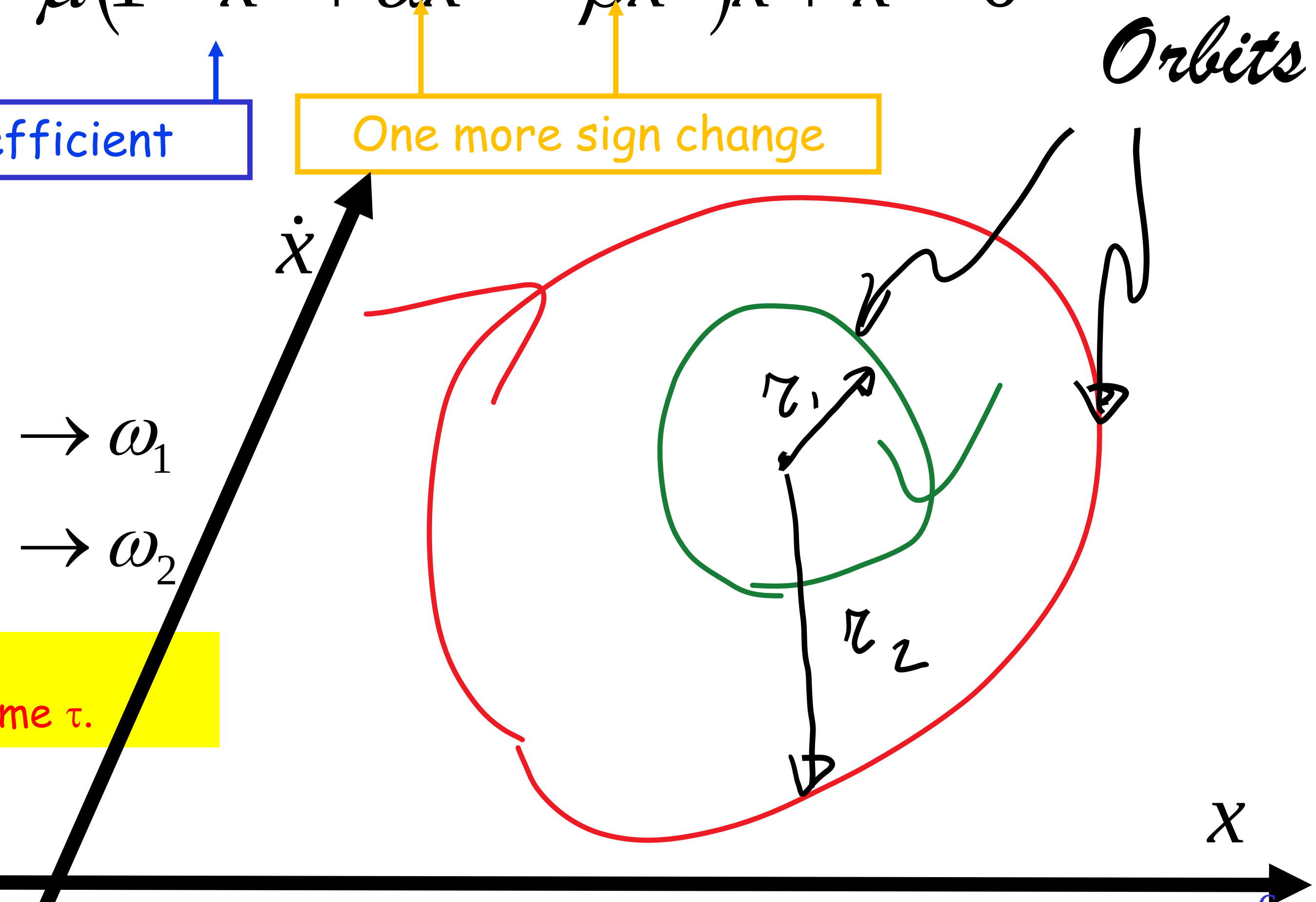


Birhythmic van der Pol oscillators:

$$\ddot{x}-\mu\left(1-x^2+\alpha x^4-\beta x^6\right)\dot{x}+x=0$$

Friction coefficient

One more sign change



Two frequencies: $\begin{cases} r_1 \rightarrow \omega_1 \\ r_2 \rightarrow \omega_2 \end{cases}$

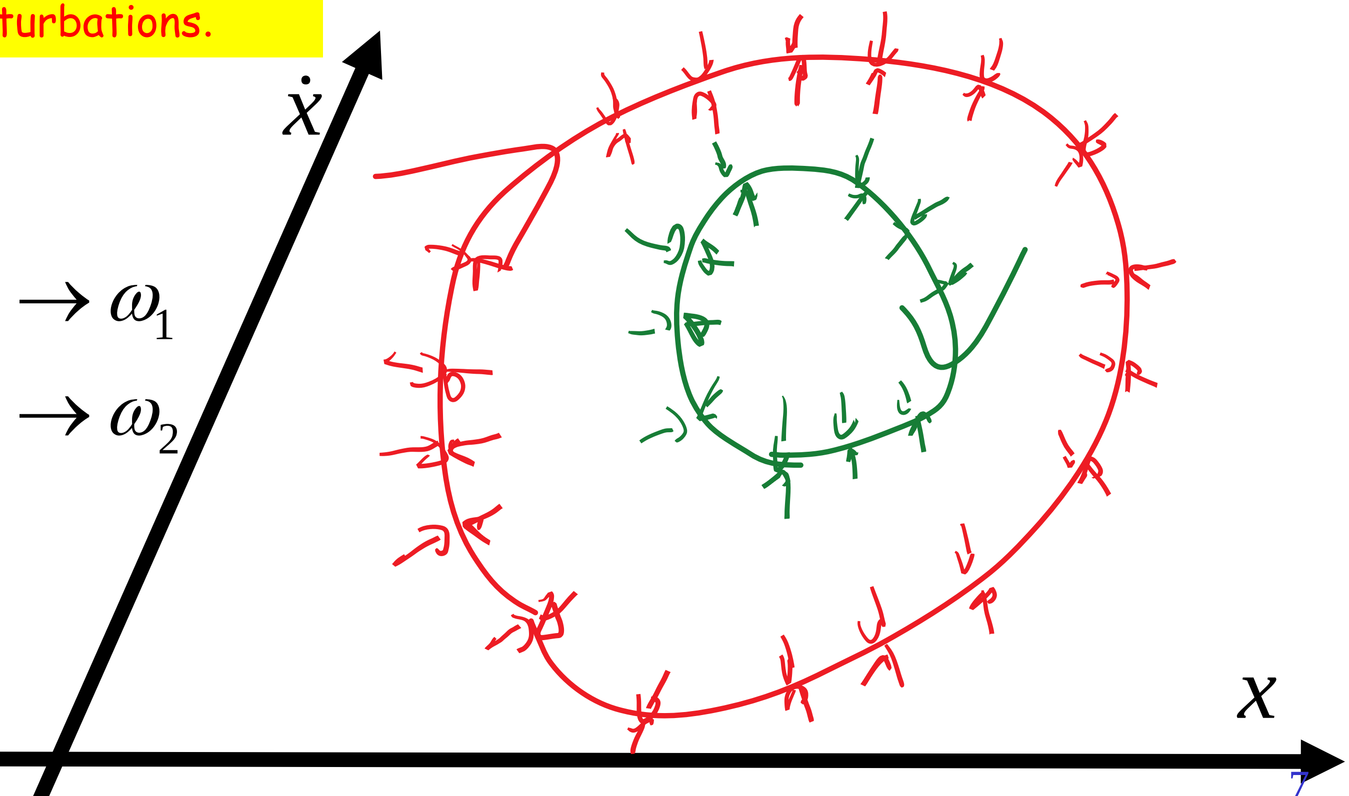
Additive noise becomes metastable with lifetime τ.

Local versus global stability

$$\ddot{x} - \mu\left(1 - x^2 + \alpha x^4 - \beta x^6\right)\dot{x} + x = 0$$

Local stability: the orbit is stable against small perturbations.

Two frequencies: $\begin{cases} r_1 \to \omega_1 \\ r_2 \to \omega_2 \end{cases}$

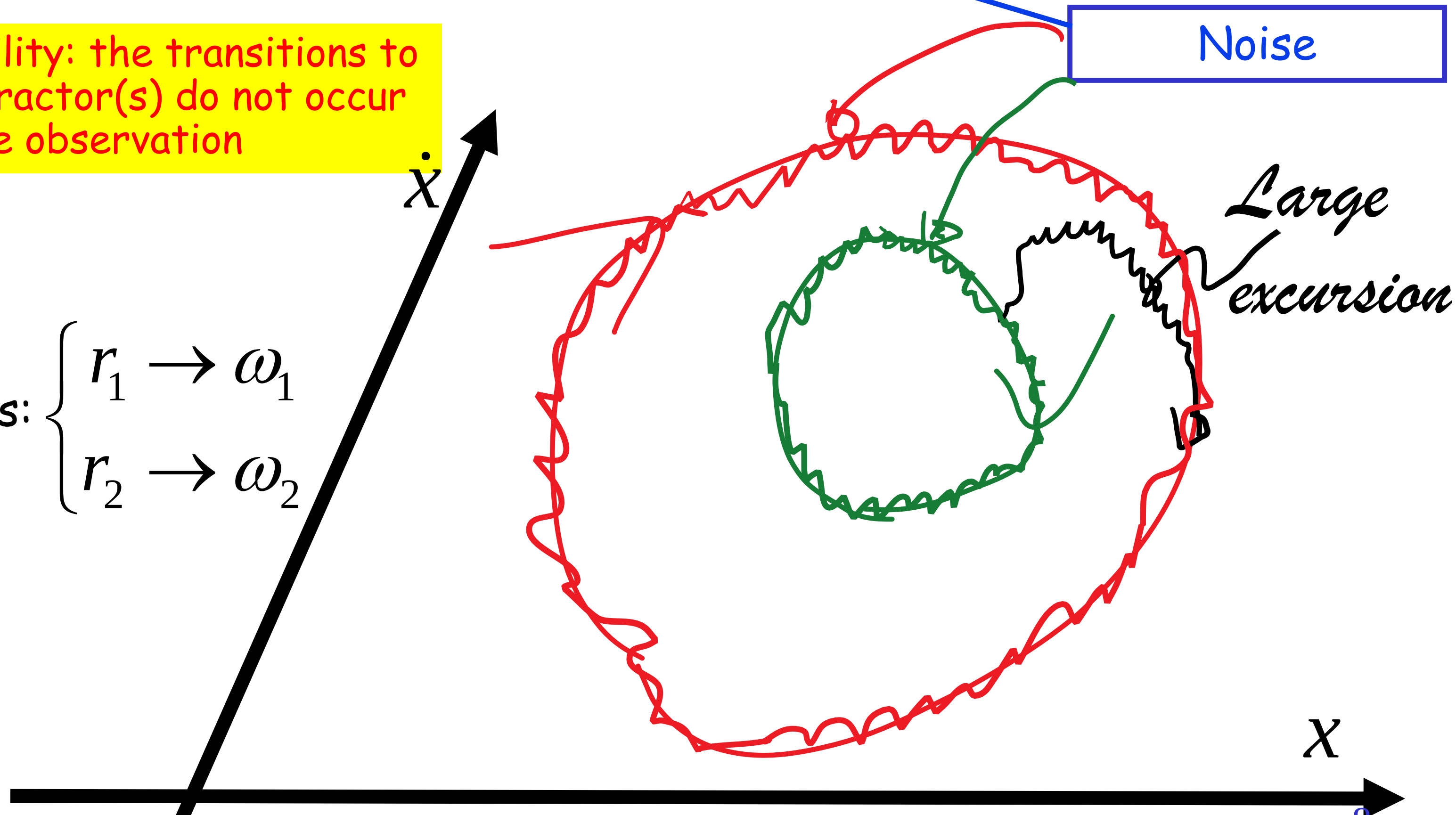


Local versus global stability

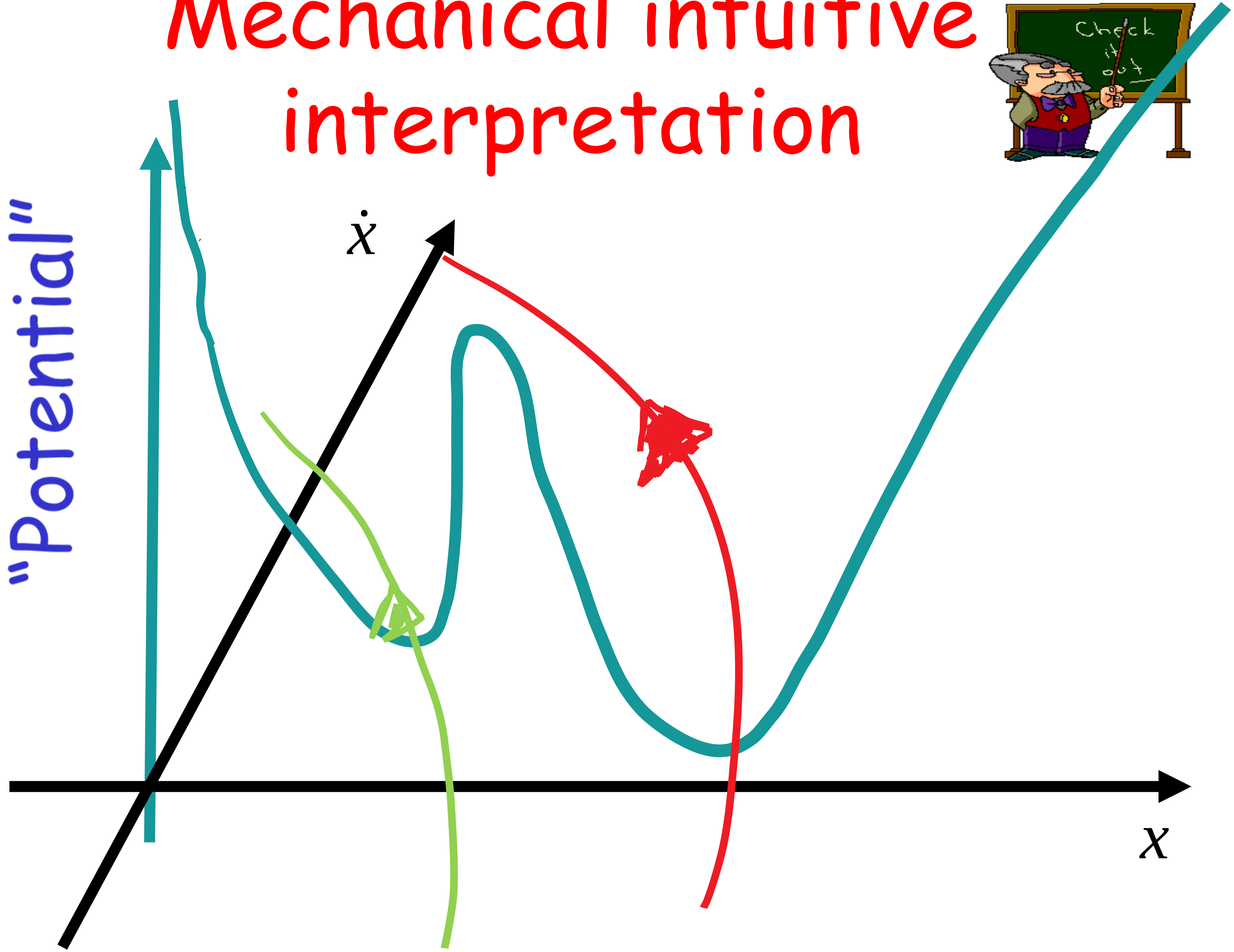
$$\ddot{x} - \mu\left(1 - x^2 + \alpha x^4 - \beta x^6\right)\dot{x} + x = \Gamma(t)$$

Global stability: the transitions to other attractor(s) do not occur during the observation

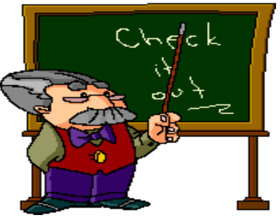
Two frequencies: $\begin{cases} r_1 \to \omega_1 \\ r_2 \to \omega_2 \end{cases}$

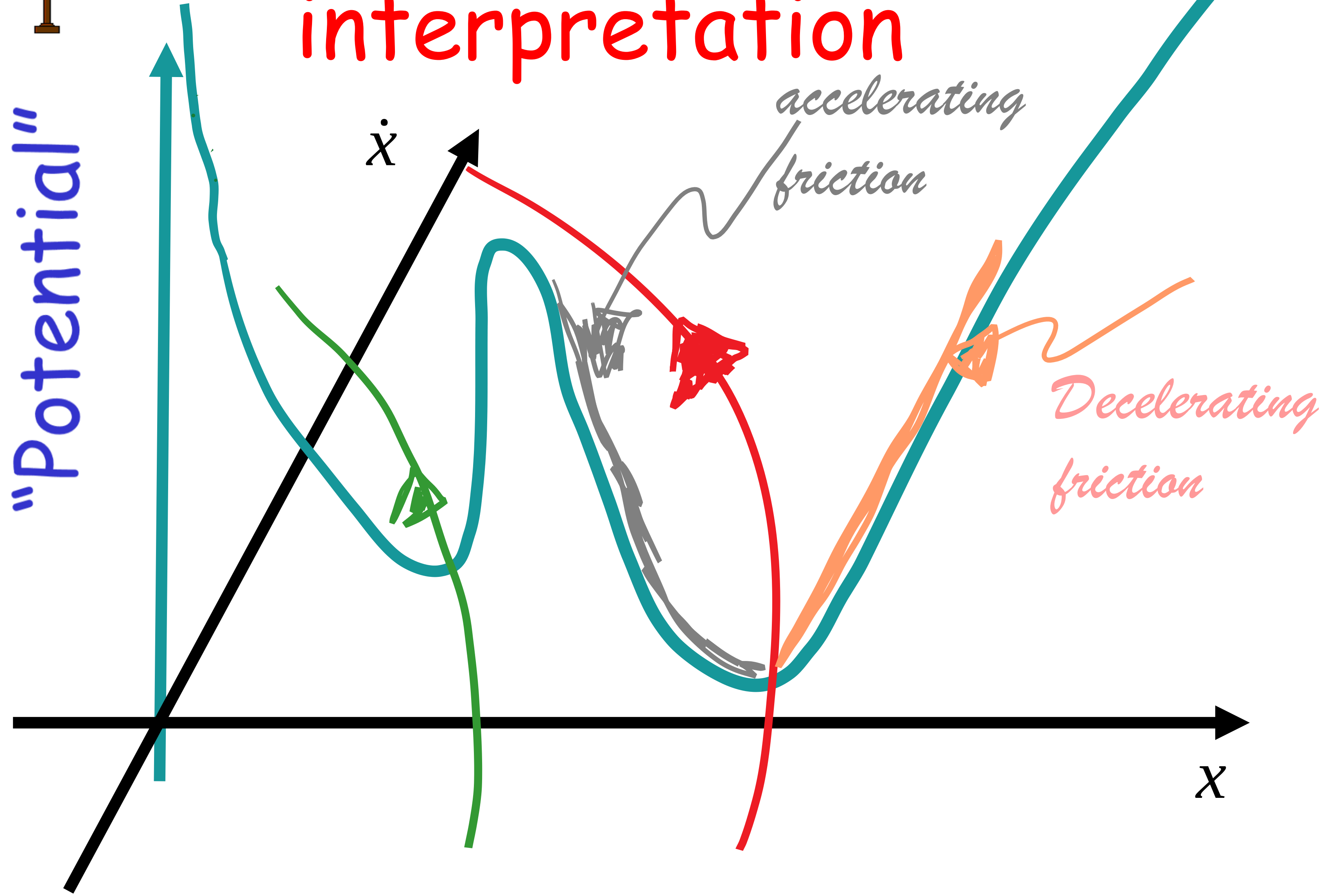


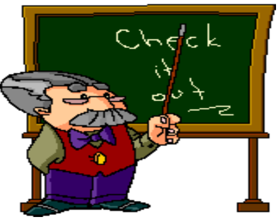
Mechanical intuitive interpretation



Mechanical intuitive interpretation

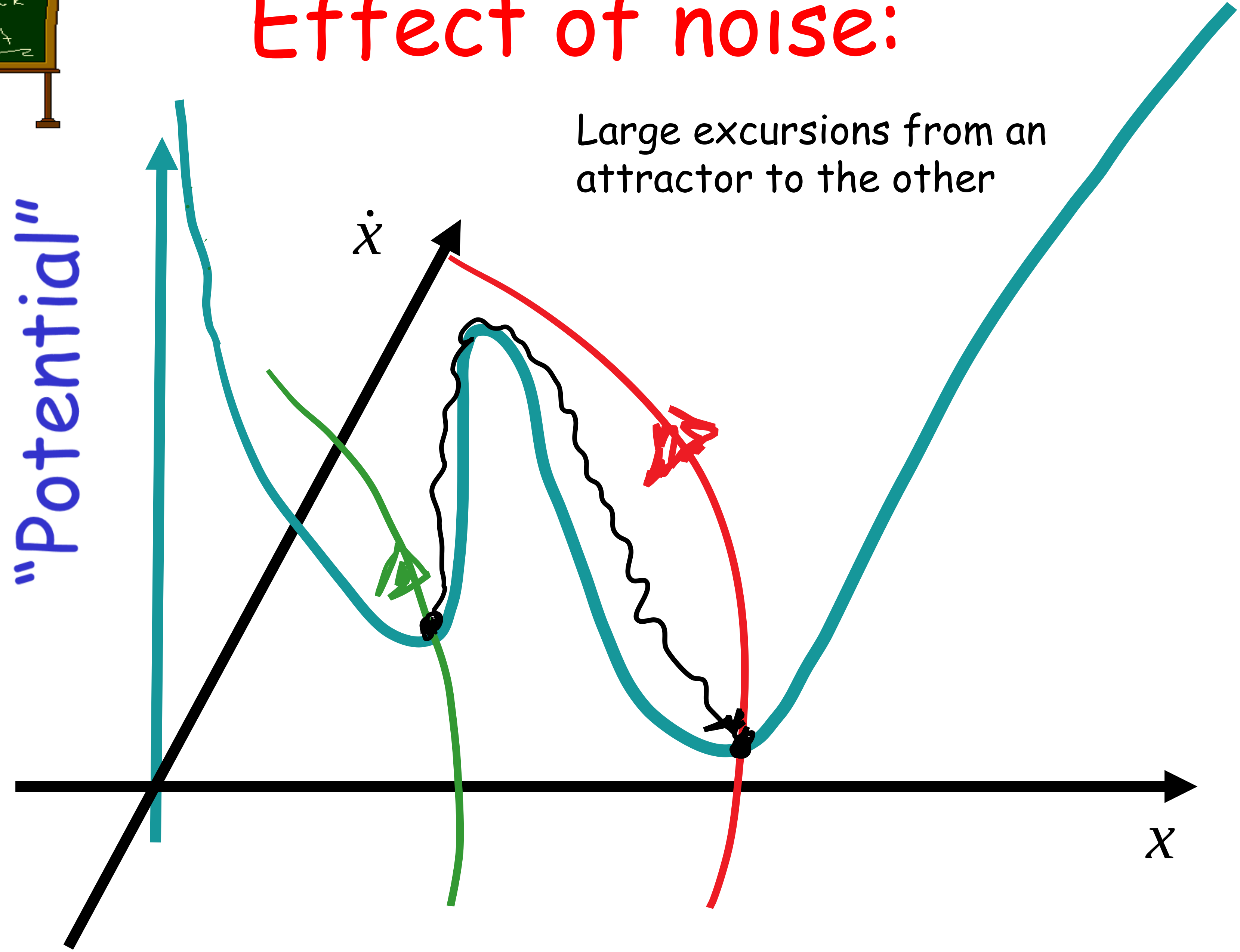






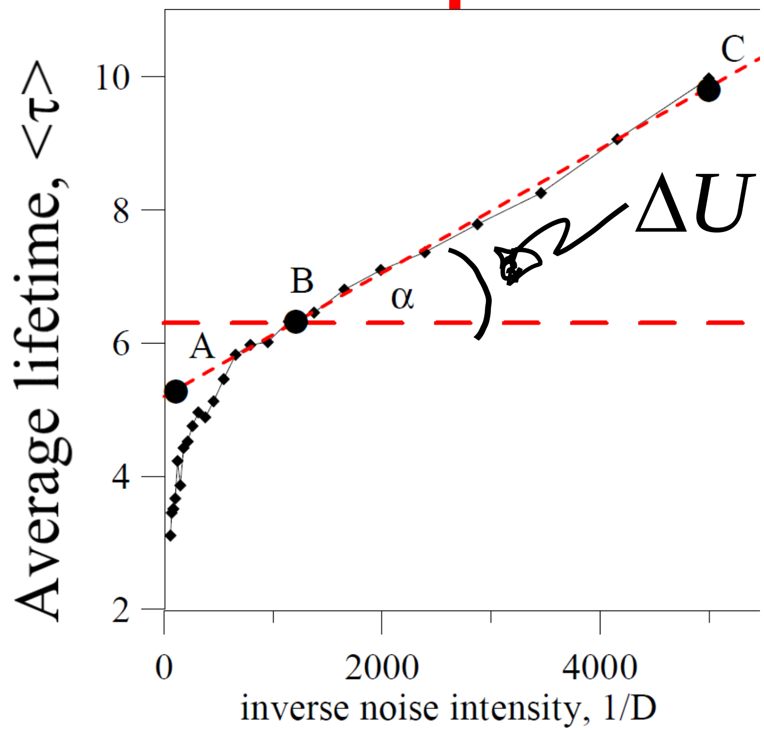
Effect of noise:

Large excursions from an attractor to the other



Numerical energy barriers through Kramers escape

$$\boxed{\langle \tau \rangle \propto e^{+\frac{\Delta U}{D}}}$$



Quasipotential: reverse the logic:

$$\Delta U \equiv \frac{\log\left[\langle \tau(D+\Delta D)\rangle\right] - \log\left[\langle \tau(D)\rangle\right]}{\Delta(1/D)}$$

Proof that it exists and it is a Lyapunov function:

[1] Dykman, M. I., Krigovlaz, M. A. Sov. Phys. JETP **50**, 30 (1979).

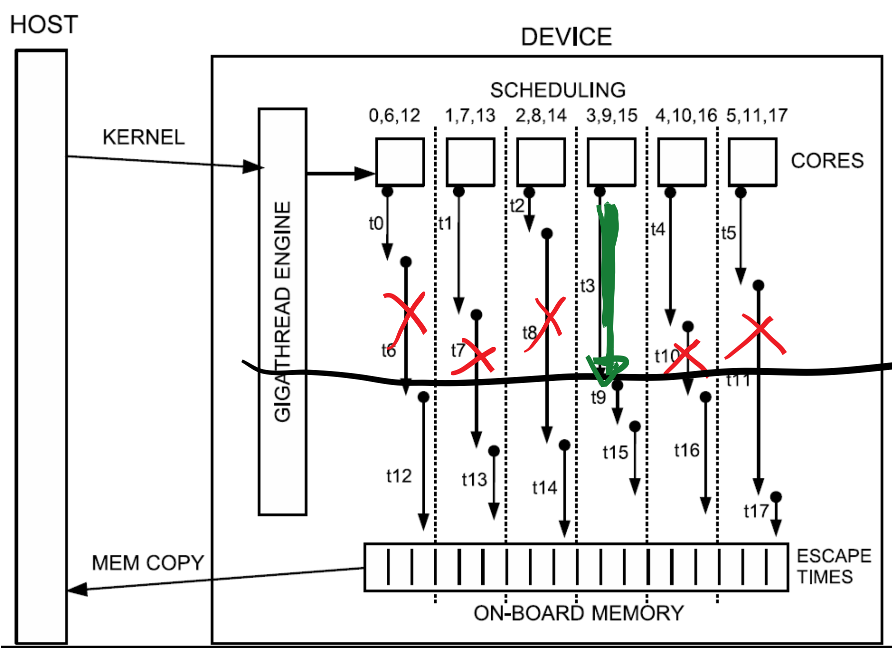
[2] R. Graham and T. Tél , *Existence of a Potential for Dissipative Dynamical Systems* Phys. Rev. Lett. **52**, 9 (1984).

[3] Graham R. Tèl, T. *Weak-noise limit of Fokker-Planck models and nondifferentiable potentials for dissipative dynamical systems*, Phys. Rev. A 31, 1109 (1985),.

Delicate point in numerical simulations

First passage time cannot be straightforwardly parallelized, for the completion occurs at random points

- Assignment of the jobs to the processors is not trivial

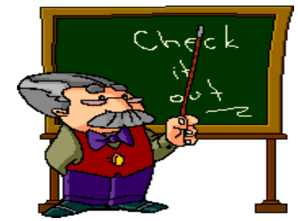


Many processors stay idle

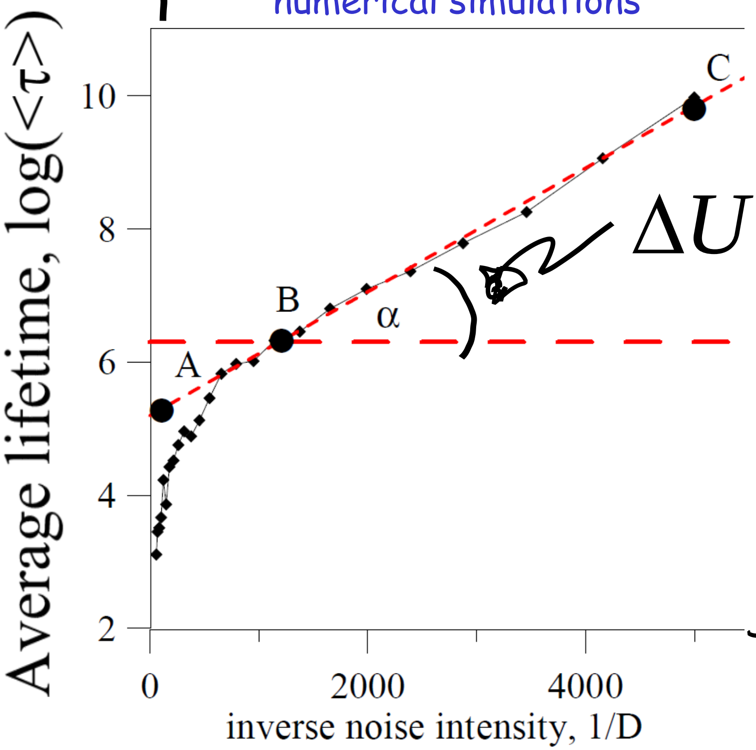
Naïve parallelization waits for the slowest escape to occur

- Pierro, Troiano, Mejuto, Filatrella, Stochastic first passage time accelerated with CUDA, J. Comput. Phys. **361**, 136 (2018).

Why it useful the concept of quasipotential



The scape time contains the information about the stability (in the form or average escape time) , the quasipotential allows to extrapolate to low noise levels not (easily) accessible with numerical simulations

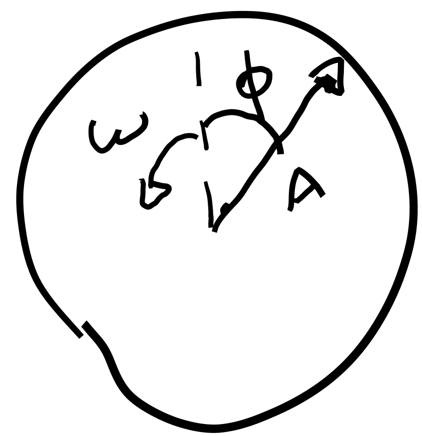


Practical application:
- superconducting devices,
- thermal disturbances in quantum computing

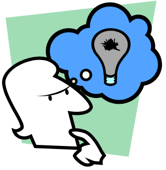
Analytical estimate through stochastic averaging

Assume the oscillations to occur with a certain frequency ω , with time dependent phase Φ and radius A - then perturb the fluctuations of the two quantities with noise

$$\begin{cases} x(t) = A(t)\cos(\omega t + \Phi(t)) \\ \dot{x}(t) = -\omega A(t)\sin(\omega t + \Phi(t)) \end{cases}$$



-Anashenko et al., Commun. Nonlinear Sci. Numer. Simulat. **30**, 15 (2016)
- Anishchenko, V.S., Astakhov, V., Neiman, A., Vadivasova, T., Schimansky-Geier, L.: Nonlinear Dynamics of Chaotic and Stochastic Systems: Tutorial and Modern Developments. Springer, Berlin (2007)

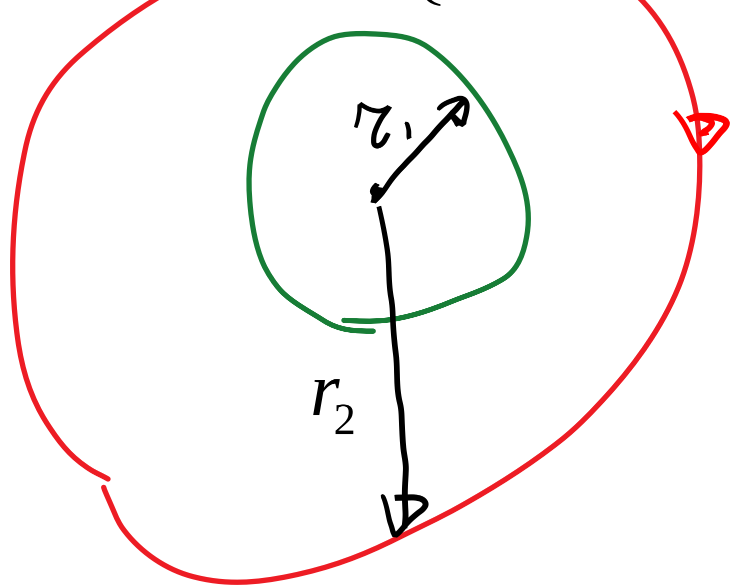


Delicate point in analytical estimate

For birhythmic systems the frequency of the approximate solution is ambiguous

$$\begin{cases} x(t) = A(t)\cos(\omega t + \Phi(t)) \\ \dot{x}(t) = -\omega A(t)\sin(\omega t + \Phi(t)) \end{cases}$$

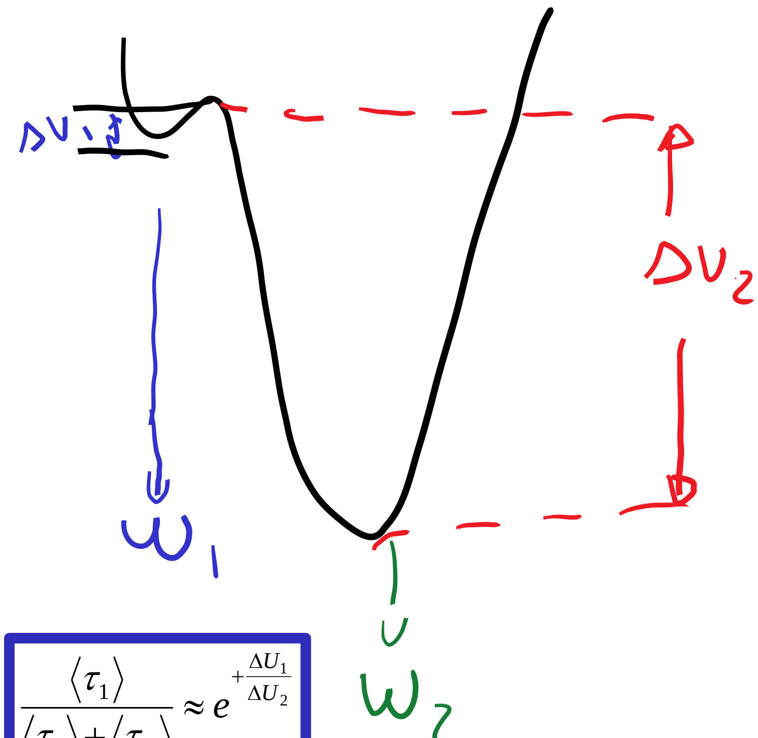
Two frequencies: $\begin{cases} r_1 \to \omega_1 \\ r_2 \to \omega_2 \end{cases}$



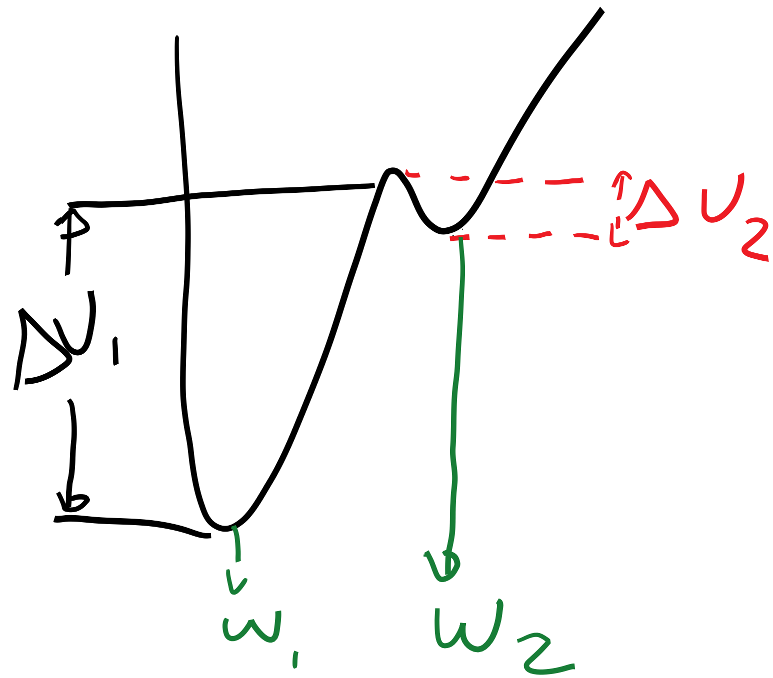
- Mbakob Yonkeu, R., Yamapi, R., Filatrella, G., Tchawoua, C (2016).*Pseudopotential of birhythmic van der Pol-type systems with correlated noise*. Nonlinear Dynamics {84}, 627; *Stochastic Bifurcations induced by correlated Noise in a Birhythmic van der Pol System*. Commun. Nonlinear Sci. Numer. Simulat. **33**, 70
- Mbakob Yonkeu, R., Yamapi, R., Filatrella, G., Tchawoua C.(2017). *Effects of a periodic drive and correlated noise on birhythmic van der Pol systems*. Physica A: **466**, 552-569.
- Yamapi, R., Filatrella, G., Aziz-Alaoui, M. A., Cerdeira, H.A. (2012)., *Effective Fokker-Planck equation for birhythmic modified van der Pol oscillator*, Chaos, **22**, 043114 .
- Yamapi, R., Chamgoué, A.C., Filatrella, G. Woafo, P. (2017). *Coherence and stochastic resonance in a birhythmic van der Pol system*. Eur. Phys. J. B. **90**, 153.
- Zakharova, A., Vadivasova, T., Anishchenko, V., Koseska, A., Kurths, J. (2010). *Stochastic bifurcations and coherencelike resonance in a self-sustained bistable noisy oscillator* . Phys. Rev. E 81, 011106.
- Ghosh P,Sen S,Riaz, SS,Ray, DS. *Controlling birhythmicity in a self-sustained oscillator by time-delayed feedback*. Phys. Rev. E (2011);**83**, 036205

Main findings - 1

The two attractors are very different =>
The residence time are exponentially different



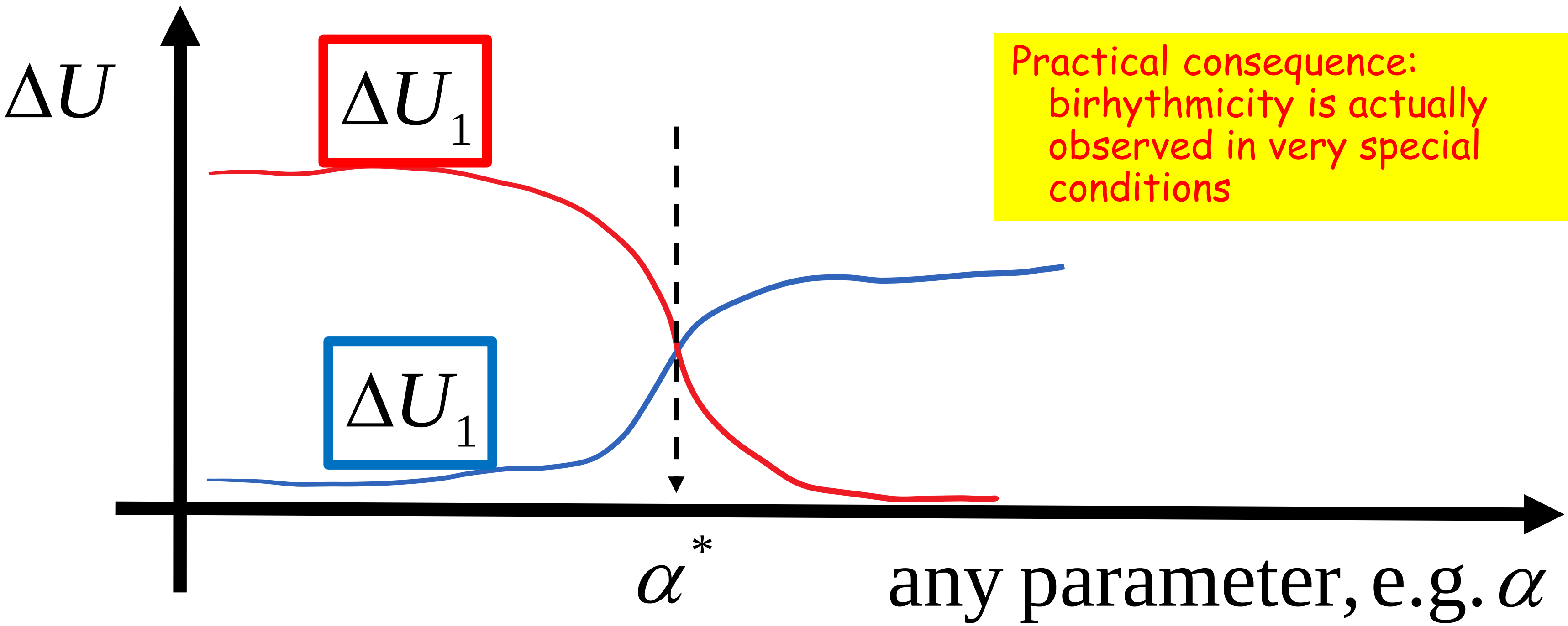
$$\frac{\langle \tau_1 \rangle}{\langle \tau_1 \rangle + \langle \tau_2 \rangle} \approx e^{+\frac{\Delta U_1}{\Delta U_2}}$$



$$\frac{\langle \tau_2 \rangle}{\langle \tau_1 \rangle + \langle \tau_2 \rangle} \approx e^{+\frac{\Delta U_2}{\Delta U_1}}$$

Main findings - 2

The transition from one dominant stable orbit to the other resembles phase transitions:



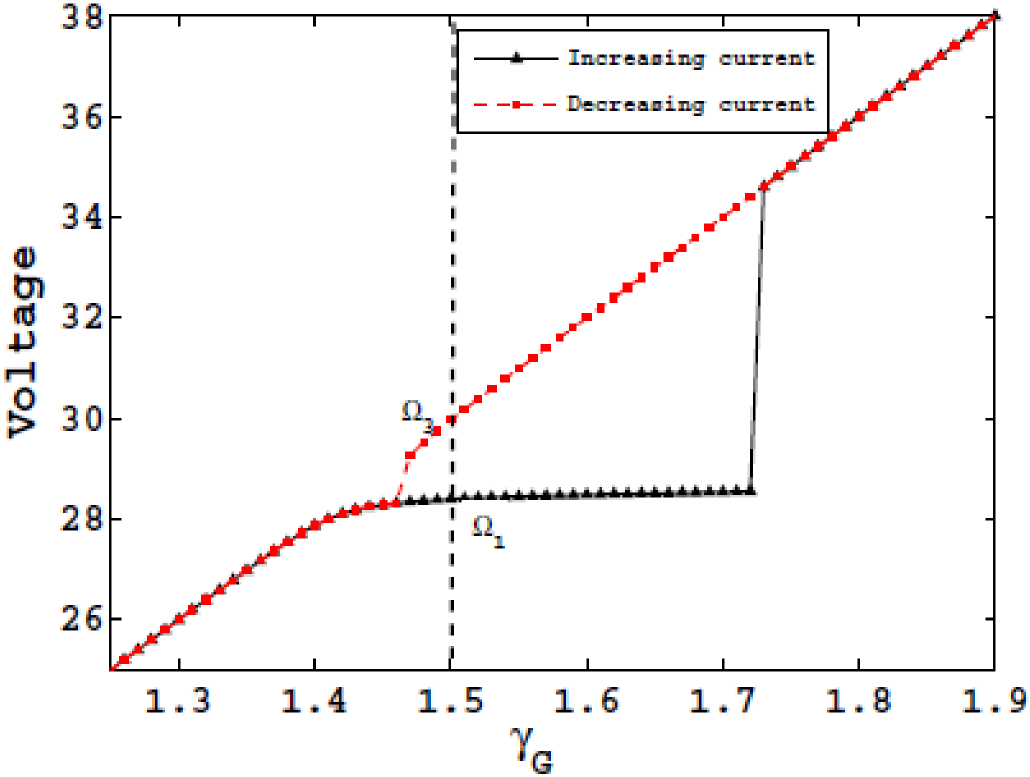
Practical consequence:
birhythmicity is actually observed in very special conditions

General feature of quasipotential:

- C. Stambaugh and H. B. Chan, Phys. Rev. B 73, 172302 2006.
- C. Stambaugh and H. B. Chan, Phys. Rev. Lett. 97, 110602 2006.

Other systems: superconducting Josephson junctions coupled to a resonator

The transition from one attractor to another depending on the initial conditions



-Yamapi, Filatrella, PRE 89, 052905 (2014); Pountougnigni, Yamapi, Filatrella, Tschawouva, Phys. Rev. E **99**, 032220 (2019)

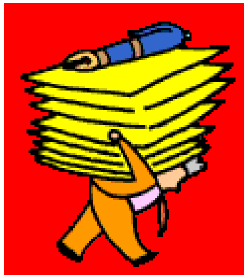
Conclusions

- Birhythmic oscillations can be modelled with a variation of the well-known van der Pol model
- The non-local or global stability can be estimated numerically and analytically, with consistent results
- The properties of the two attractors can be very different, and therefore only one is effectively stable (statistically visited the overwhelming time)
- The passage from a dominant attractor to the other occurs very fast, in analogy with phase transitions

Perspectives

- Experimental data on birhythmic systems
- Non Gaussian noise

One sentence conclusion



The stability against noise of attractors in non-potential s[illegible]s can be ascertain with the h[illegible]he quasi-pot[illegible]

Thanks for your attention