periodicity, delays and numerical methods in biomathematics: a recent account

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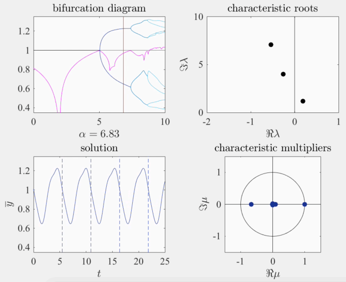
overview

- periodicity: seasonality, recurrences, chaos
- delays: Mackey-Glass [1]

$$y'(t) = \frac{2y(t-2)}{1 + y(t-2)^{\alpha}} - y(t)$$



- numerical methods:
  - periodic solutions, by collocation of BVPs [2]
  - Floquet multipliers, by collocation of monodromy operators [3]
  - consolidated for delay differential equations: `DDE-Biftool` (`AUTO`, `MatCont`)

[1] Mackey, Glass – Science 1977
[2] Engelborghs, Luzyanina, In 't Hout, Roose – SISC 2001
[3] B., Maset, Vermiglio – SINUM 2012

coupled equations

$$x(t) = F(x_t, y_t) \tag{RE}$$
$$y'(t) = G(x_t, y_t) \tag{DDE}$$

- $x_t \in X := L^1([-\tau, 0]; \mathbb{R}^{d_X})$, $x_t(\theta) := x(t+\theta)$, $\theta \in [-\tau, 0]$
- $y_t \in Y := C([-\tau, 0]; \mathbb{R}^{d_Y})$, defined as x_t
- $F : X \times Y \to \mathbb{R}^{d_X}$, $G : X \times Y \to \mathbb{R}^{d_Y}$ nonlinear, smooth, F integral in X

- interest in realistic models (e.g., *Daphnia* [1])

[1] Diekmann, Gyllenberg, Metz, Nakaoka, de Roos – JMB 2010

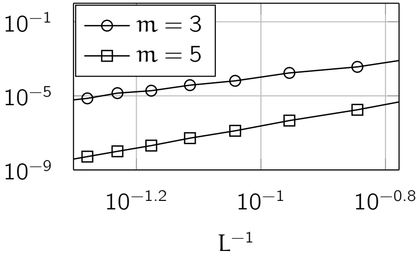
references

- preliminary work on renewal equations:
  - B., Diekmann, Liessi and Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ., 65:1–24, 2016
- computation of Floquet multipliers:
  - Liessi, *Pseudospectral methods for the stability of periodic solutions of delay models*, PhD thesis, University of Udine, 2018
  - B. and Liessi, *Approximation of eigenvalues of evolution operators for linear renewal equations*, SIAM J. Numer. Anal., 56(3):1456–1481, 2018
  - B. and Liessi, *Approximation of eigenvalues of evolution operators for linear coupled renewal and retarded functional differential equations*, submitted
- computation of periodic solutions:
  - Andò, *Collocation methods for complex delay models of structured populations*, PhD thesis, University of Udine, 2020
  - Andò and B., *Convergence analysis of collocation methods for computing periodic solutions of retarded functional differential equations*, submitted

facts

- extend piecewise collocation [1] (m-degree polynomials, L intervals):

$$\begin{cases} x(t) = \dfrac{\gamma}{2}\displaystyle\int_{-3}^{-1} x(t+\theta)[1-x(t+\theta)]\,d\theta \\ y'(t) = \gamma x(t)\{x(t-1)[1-x(t-1)] - x(t-3)[1-x(t-3)]\} + y(t) \end{cases}$$



- lack of a general proof of convergence due to unknown period (even for DDEs)
- [3] inspired by [2]

[1] Engelborghs, Luzyanina, In 't Hout, Roose – SISC 2001
[2] Maset – NM 2016, SINUM 2015 I and II
[3] Andò, B. – submitted

time scaling and BVPs

- for $G : Y \to \mathbb{R}^d$ defined on a state space Y of functions $[-\tau, 0] \to \mathbb{R}^d$ let
$$y'(t) = G(y_t)$$
have an ω-periodic solution $(\omega \geqslant \tau)$

- as ω is unknown, let $t = s_\omega(t) := t/\omega$ to get a 1-periodic solution of
$$y'(t) = \omega G(y_t \circ s_\omega)$$

- equivalent BVPs:

classic, $y \in [0, 1]$:	Halanay, $y \in [-1, 1]$:
$\begin{cases} y'(t) = \omega G(\overline{y_t} \circ s_\omega), \ t \in [0, 1], \\ y(0) = y(1) \\ p(y) = 0 \end{cases}$ $\overline{v}(\sigma) := \begin{cases} v(\sigma), & \sigma \in [0, 1], \\ v(\sigma + 1), & \sigma \in [-1, 0), \end{cases}$	$\begin{cases} y'(t) = \omega G(y_t \circ s_\omega), \ t \in [0, 1], \\ y_0 = y_1 \\ p(y\vert_{[0,1]}) = 0 \end{cases}$

(p suitable phase condition)

from BVP to fixed point

recast

$$\begin{cases} y'(t) = \omega G(y_t \circ s_\omega), \ t \in [0,1], \\ y_0 = y_1 \\ p(y|_{[0,1]}) = 0 \end{cases}$$

as follows by choosing suitable spaces of functions

- in $[0,1]$ for the derivative ($\mathbb{U}$)
- in $[-1,1]$ for the solution ($\mathbb{V}$)
- in $[-1,0]$ for the initial function ($\mathbb{A}$)

find $(v^*, \omega^*) \in \mathbb{V} \times \mathbb{R}$ with $v^* = \mathcal{G}(u^*, \psi^*)$ for

$$\mathcal{G} : \mathbb{U} \times \mathbb{A} \to \mathbb{V}, \qquad \mathcal{G}(u,\psi)(t) := \begin{cases} \psi(0) + \displaystyle\int_0^t u(s)\,ds, & t \in [0,1], \\ \psi(t), & t \in [-1,0], \end{cases}$$

and $(u^*, \psi^*, \omega^*) \in \mathbb{U} \times \mathbb{A} \times \mathbb{R}$ a fixed point of

$$\Phi : \mathbb{U} \times \mathbb{A} \times \mathbb{R} \to \mathbb{U} \times \mathbb{A} \times \mathbb{R}, \qquad \Phi(u,\psi,\omega) := \begin{pmatrix} \omega G(\mathcal{G}(u,\psi)_\cdot \circ s_\omega) \\ \mathcal{G}(u,\psi)_1 \\ \omega - p(\mathcal{G}(u,\psi)|_{[0,1]}) \end{pmatrix}$$

collocation

- set $X := \mathbb{U} \times \mathbb{A} \times \mathbb{R}$
- define $X_L := \mathbb{U}_L \times \mathbb{A}_L \times \mathbb{R}$ based on piecewise collocation with m-degree polynomials on L intervals via
  - restriction $R_L : X \to X_L$
  - prolongation $P_L : X_L \to X$
  - projection $\Pi_L := P_L R_L : X \to X$
- use
  - $R_L \Phi P_L : X_L \to X_L$ for computation
  - $\Pi_L \Phi : X \to X$ for convergence ($L \to \infty$, fixed m)

stability, consistency and smoothness

- assume existence of isolated fixed point x^* of Φ:
  - Φ Fréchet differentiable
  - $D\Phi$ locally Lipschitz at x^*
  - $[I_X - D\Phi(x^*)]^{-1}$ exists bounded
- compare exact and discrete problems

$$x^* = \Phi(x^*), \qquad x_L^* = \Pi_L \Phi(x_L^*)$$

to get

$$[I_X - \Pi_L D\Phi(x^*)](x_L^* - x^*) = \Pi_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)] + \Pi_L x^* - x^*$$

through

$$\begin{aligned} x_L^* - x^* &= \Pi_L \Phi(x_L^*) - \Phi(x^*) \\ &= \Pi_L[\Phi(x_L^*) - \Phi(x^*)] + [\Pi_L \Phi(x^*) - x^*] \\ &= \Pi_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)] + \Pi_L D\Phi(x^*)(x_L^* - x^*) \\ &\quad + \Pi_L x^* - x^* \end{aligned}$$

- existence and uniqueness of x_L^* and convergence of x_L^* to x^* ask for
  - $[I_X - \Pi_L D\Phi(x^*)]^{-1}$ exists uniformly bounded (stability)
  - $\|\Pi_L x^* - x^*\|_X \to 0$ (consistency)
  - $\|\Pi_L[\Phi(x_L^*) - \Phi(x^*) - D\Phi(x^*)(x_L^* - x^*)]\|_X \leqslant k\|x_L^* - x^*\|_X$ uniformly, k small

role of periodicity

$$\Phi(u,\psi,\omega) := \begin{pmatrix} \omega G(\mathcal{G}(u,\psi). \circ s_\omega) \\ \mathcal{G}(u,\psi)_1 \\ \omega - p(\mathcal{G}(u,\psi)|_{[0,1]}) \end{pmatrix}$$

- Φ Fréchet differentiable:
  - $\partial_\omega G(\mathcal{G}(u,\psi). \circ s_\omega)$ asks for some regularity in $\mathbb{V}$, but not too much (not C^1)
  - use right-hand derivatives (correct with delays)
- failure of classic BVP, for which

$$\mathcal{G}(u,\alpha)(t) := \alpha + \int_0^t u(s)\,ds, \qquad t \in [0,1],\ \alpha \in \mathbb{R}$$

  - $D\Phi$ locally Lipschitz at x^*: $\overline{\mathcal{G}(u,\alpha).}$ not even continuous, unless

$$\int_0^1 u(s)\,ds = 0$$

  - $[I_X - D\Phi(x^*)]^{-1}$ exists bounded: given any $\alpha_0 \in \mathbb{R}$, find $\alpha \in \mathbb{R}$ such that

$$\alpha = \mathcal{G}(u,\alpha)(1) + \alpha_0,$$

  impossible if u as above

error and convergence

Theorem. Let $x^* = (u^*, \psi^*, \omega^*)$ be an isolated fixed point of Φ with Φ Fréchet differentiable, $D\Phi$ locally Lipschitz at x^* and let $[I_X - D\Phi(x^*)]^{-1}$ exist bounded. Then, for sufficiently large L, $R_L \Phi P_L$ has a unique fixed point $x_L^* = (u_L^*, \psi_L^*, \omega_L^*)$ and

$$\|(v_L^*, \omega_L^*) - (v^*, \omega^*)\|_{\mathbb{V}\times\mathbb{R}} \leqslant \kappa \|\Pi_L x^* - x^*\|_X$$

holds for $v_L^* := \mathcal{G}(\pi_L^+ u_L^*, \pi_L^- \psi_L^*)$ and κ constant ($\pi_L^\pm$ interpolant).

Theorem. Let $G \in C^q(Y, \mathbb{R}^d)$ for some integer $q \geqslant 1$. Then $u^* \in C^q([0,1], \mathbb{R}^d)$, $\psi^* \in C^{q+1}([-1,0], \mathbb{R}^d)$, $v^* \in C^{q+1}([-1,1], \mathbb{R}^d)$ and

$$\|\Pi_L x^* - x^*\|_X = O\left(L^{-\min\{m,q\}}\right).$$

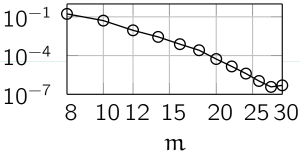
- one order less than [1]: the Halanay formulation requires to discretize also $\mathbb{A}$
- same order as (experimented) in the literature of periodic BVPs

[1] Maset – SINUM 2015 I

next

- convergence as $m \to \infty$ for fixed L not proved, yet

$$y'(t) = y(t)[\lambda - y(t-1)]$$



- proofs for REs and REs/DDEs

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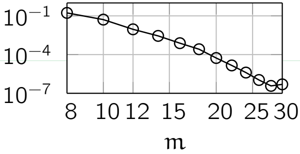

- proofs for REs and REs/DDEs

- thanks to
  - S. Maset (Trieste), J. Sieber (Exeter), D. Barton (Bristol)

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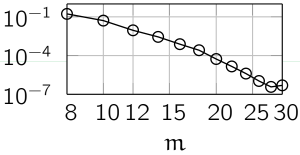

- proofs for REs and REs/DDEs

- thanks to
  – S. Maset (Trieste), J. Sieber (Exeter), D. Barton (Bristol)
  – you all for the attention!