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Inferring network structure from oscillating systems with cointegrated phase processes

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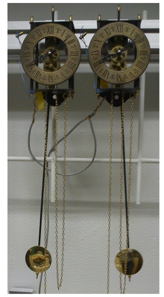
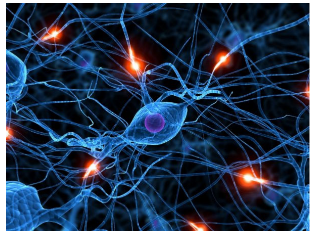
Motivation: Synchronization in Nature

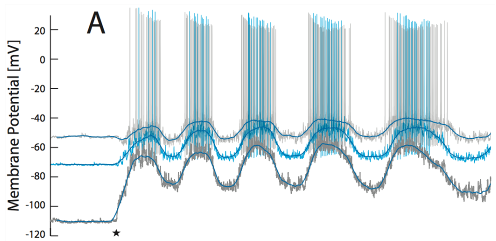




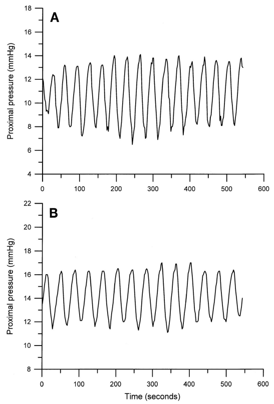
▸ Metronomes ▸ Millenium bridge in London

Motivation

Membrane potential in spinal motoneurons in a turtle



Tubuloglomerular pressure in nephrons in a rat kidney



Oscillators: A Definition

Studying biological rythms corresponds to studying systems of periodic processes. What does *periodic* mean in a stochastic system?

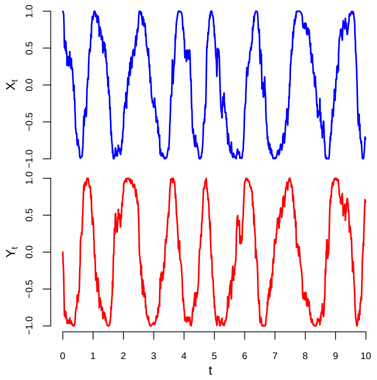
Consider a process in polar coordinates $(\phi_t, \gamma_t)'$, where $\phi_t \in \mathbb{R}$ is the *phase* process and $\gamma_t \in \mathbb{R}_+$ is the *amplitude* process, such that

$$x_t = \gamma_t \cos(\phi_t)$$
$$y_t = \gamma_t \sin(\phi_t)$$

We then define the process $z_t = (x_t, y_t)' \in \mathbb{R}^2$, $t \in [0, \infty)$ to be an oscillator if the phase process has a monotonic trend.

Oscillators: A Definition

Assume a bivariate process $z_t = (x_t, y_t)'$, such that we observe something like this



We define the *phase process* $\phi_t \in \mathbb{R}$ through the SDE

$$d\phi_t = \mu_t dt + \sigma dW_t$$

and

$$x_t = \gamma_t \cos(\phi_t)$$
$$y_t = \gamma_t \sin(\phi_t),$$

for some non-negative amplitude process γ_t.

Oscillators: Multivariate Phase Process

p oscillators, phase/amplitude-processes $\phi_t, \gamma_t \in \mathbb{R}^p$:

$$d\phi_t = f(\phi_t, \gamma_t)dt + \Sigma_\phi dW_t^\phi$$
$$d\gamma_t = g(\phi_t, \gamma_t)dt + \Sigma_\gamma dW_t^\gamma$$

Assume $\gamma_t \in \mathbb{R}_+^p$ and $\mathbb{E}[\phi_{kt}]$ strictly monotonic in t, with $a < \mathbb{E}[\phi_{k(t+1)}] - \mathbb{E}[\phi_{kt}] < A$ for some $a, A > 0$, all t, k.

For Σ's diagonal, Itôs formula yields

$$d\begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(\sigma_k^\phi)^2 & -f_k(\phi_t, \gamma_t) \\ f_k(\phi_t, \gamma_t) & -\frac{1}{2}(\sigma_k^\phi)^2 \end{pmatrix} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dt + \begin{pmatrix} 0 & -\sigma_k^\phi \\ \sigma_k^\phi & 0 \end{pmatrix} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dW_{kt}^\phi$$
$$+ \frac{g_k(\phi_t, \gamma_t) + \sigma_k^\gamma \sigma_k^\phi}{\gamma_{kt}} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dt + \frac{\sigma_k^\gamma}{\gamma_{kt}} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dW_{kt}^\gamma.$$

Covers classical oscillators like FitzHugh-Nagumo, van der Pol, Duffing...

Oscillators: Multivariate Phase Process

Assume the amplitude process $\gamma_t = \gamma > 0$ constant. Model:

$$d\phi_t = f(\phi_t)dt + \Sigma dW_t \qquad (1)$$

Compare with the Kuramoto model, a classical model of coupled phases. For $i = 1, \ldots, p$:

$$d\phi_{it} = \Big(\frac{\alpha_i}{p} \sum_{j=1}^{p} \sin(\phi_{jt} - \phi_{it}) + \mu_i\Big)dt + \sigma_i dW_{it}$$

- Eq. (1) covers Kuramoto!
- We must restrict to linear $f(\phi_t) = \Pi\phi_t + \mu_t$ for now...

Linear phase coupling

$$f_k(\phi_t) = \sum_{j=1}^{p} \Pi_{kj}(\phi_{jt} - \omega_j), \qquad k = 1, \ldots, p$$

for $\Pi \in \mathbb{R}^{p \times p}$ and $\omega = (\omega_1, \ldots, \omega_p)' \in \mathbb{R}^p$.

The interactions between oscillators are given by Π. Diagonal Π: No coupling.

The *coupling strength*: absolute values of entries of Π. The coupling has a direction!

Row k of Π defines how oscillator k depends on the rest.

Note: ω is the attracting state for the phase relations

Linear phase coupling

Let γ_t be constant, then

$$d\begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sigma_k^2 & -f_k(\phi_t) \\ f_k(\phi_t) & -\frac{1}{2}\sigma_k^2 \end{pmatrix}\begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dt + \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dW_k,$$

where $f_k(\phi_t) = \sum_j \Pi_{kj}\phi_{jt} + \mu_k$.

If $\Pi = 0$ (or diagonal) then p independent oscillators.

Eigenvalues of the deterministic drift matrix are $-\frac{\sigma^2}{2} \pm i\mu$, implying that the solutions oscillate for $\mu \neq 0$.

The oscillations are damped by the negative real part, but sustained by the noise term.

Cointegration: Unit Roots

Consider a p-dimensional autoregressive process

$$x_t = Ax_{t-1} + \varepsilon_t,$$

where $A \in \mathbb{R}^{p\times p}$ and $\varepsilon_t \in \mathbb{R}^p$.

The associated characteristic polynomial for x_t is

$$C(z) = \det(I_p - Az), \quad z \in \mathbb{C}.$$

If $C(z) \neq 0$ for $|z| \leq 1$ the process is *stationary*.

If $C(z) = 0$ for $z = 1$, the process is *nonstationary*.

A process with a unit root of multiplicity d is *integrated of order d*: $I(d)$, and we say that x_t has a *stochastic trend*.

For x_t an $I(d)$ process: Δx_t is $I(d-1)$, $\Delta^d x_t$ is $I(0)$.

Cointegration: Vector Error Correction

Assume $x_t \in \mathbb{R}^p$ is $I(1)$ and rewrite

$$x_t = Ax_{t-1} + \varepsilon_t,$$

as

$$\Delta x_t = \Pi x_{t-1} + \varepsilon_t,$$

where

$$\Pi = -(I_p - A).$$

If x_t is $I(1)$ then Δx_t is $I(0)$ and thus Πx_{t-1} must be $I(0)$.

Cointegration: Vector Error Correction

3 possibilities for $\text{rank}(\Pi) = r$:

- Π has full rank p.
- Π has reduced rank $0 < r < p$.
- Π has rank 0.

$r = p \Rightarrow$ then x_t *must* be $I(0)$.

$r = 0 \Rightarrow$ no stationary relations of x_t.

$0 < r < p \Rightarrow r$ stationary combinations of x_t variables. We then say that x_t is a *cointegrated* process.

Cointegration: Parameters

If Π has rank $0 < r < p$, then

$$\Pi = \alpha\beta',$$

where $\alpha, \beta \in \mathbb{R}^{p \times r}$ and rank $r < p$.

We then have

$$(\alpha'\alpha)^{-1}\alpha'\Pi x_t = (\alpha'\alpha)^{-1}\alpha'\alpha\beta' x_t = \beta' x_t$$

is $I(0)$.

Hence the r linearly independent columns of β correspond to r stationary linear combinations of x_t.

Note also that α and β are not uniquely identified!

Solved by normalizing: $\hat{\beta} = \begin{pmatrix} I_r \\ \tilde{\beta}_{p-r,r} \end{pmatrix}$

Estimation

Assume the amplitude process $\gamma_t = \gamma > 0$ constant.
Model:

$$d\phi_t = (\Pi\phi_t + \mu)dt + \Sigma dW_t$$

First the rank is estimated through a series of likelihood ratios tests (Johansen's test).

Then a reduced rank regression with least squares is performed.

Simulation: Winfree oscillator

$$\begin{aligned} d\gamma_{kt} &= (\kappa_k - \gamma_{kt})\gamma_{kt}^2 dt + \sigma_k^\gamma dW_{kt}^\gamma \\ d\phi_{kt} &= \left(\sum_{j=1}^{p} \Pi_{kj}\phi_j + \gamma_{kt}\right)dt + \sigma_k^\phi dW_{kt}^\phi. \end{aligned}$$

By Itôs formula:

$$d\begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} =$$

$$\begin{pmatrix} (\kappa_k - \gamma_{kt})\gamma_{kt} + \gamma_{kt}^{-1}\sigma_k^\gamma\sigma_k^\phi - \frac{(\sigma_k^\phi)^2}{2} & -\left(\sum_{j=1}^{p} \Pi_{kj}\phi_j + \gamma_{kt}\right) \\ \left(\sum_{j=1}^{p} \Pi_{kj}\phi_j + \gamma_{kt}\right) & (\kappa_k - \gamma_{kt})\gamma_{kt} + \gamma_{kt}^{-1}\sigma_k^\gamma\sigma_k^\phi - \frac{(\sigma_k^\phi)^2}{2} \end{pmatrix} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dt$$

$$+ \begin{pmatrix} 0 & -\sigma_k^\phi \\ \sigma_k^\phi & 0 \end{pmatrix} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dW_{kt}^\phi + \begin{pmatrix} \gamma_{kt}^{-1}\sigma_k^\gamma & 0 \\ 0 & \gamma_{kt}^{-1}\sigma_k^\gamma \end{pmatrix} \begin{pmatrix} x_{kt} \\ y_{kt} \end{pmatrix} dW_{kt}^\gamma$$

Simulation: Overall Setup

Shared parameters:

$$\begin{aligned} p &= 3 \\ \sigma_1^\phi &= \sigma_2^\phi = \sigma_3^\phi = 1 \\ \sigma_1^\gamma &= \sigma_2^\gamma = \sigma_3^\gamma = 0.1 \\ \kappa &= (0.75, 1, 1)' \\ z_0 &= (1, 0, 0, 1, -1, 0)' \end{aligned}$$

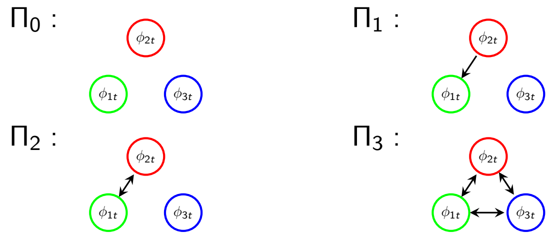
Simulate 100k steps with $\tilde{\Delta}t = 0.001$ using Euler-Maruyama, then subsample every 100th to obtain 1000 observations with timestep $\Delta t = 0.1$, i.e. $t \in [0, 100]$.

Simulation: Models

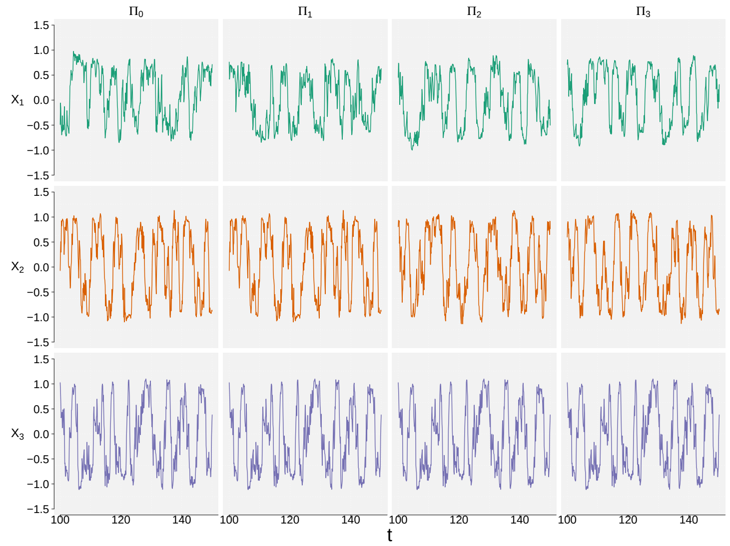
Four different systems

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

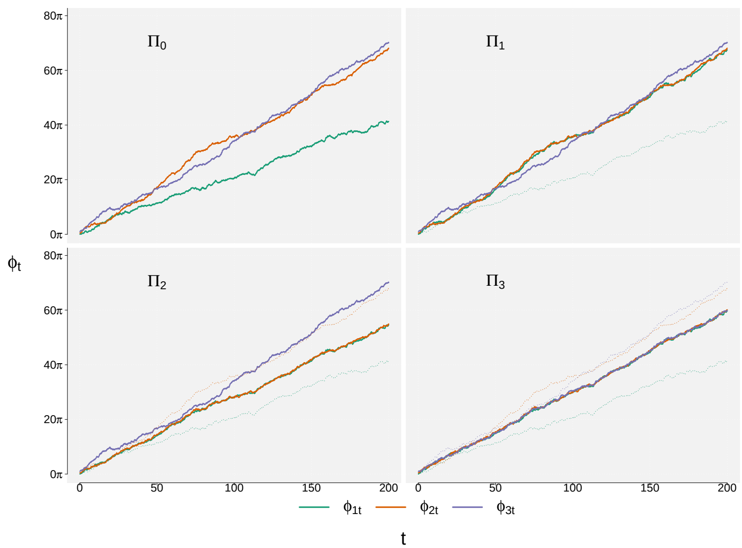
$$\Pi_2 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Pi_3 = \begin{pmatrix} -0.5 & 0.25 & 0.25 \\ 0.25 & -0.5 & 0.25 \\ 0.25 & 0.25 & -0.5 \end{pmatrix}$$



Simulation: Trajectories



Simulation: Phases



Simulation: Rank tests for rank(Π)

Johansen rank tests.

Model	H_r	Test values	p-value
Π_0	$r = 0$	14.94	0.751
	$r \leq 1$	6.73	0.519
	$r \leq 2$	0.17	0.635
Π_1	$r = 0$	52.50	0.000
	$r \leq 1$	5.61	0.489
	$r \leq 2$	0.78	0.306
Π_2	$r = 0$	64.78	0.000
	$r \leq 1$	6.57	0.305
	$r \leq 2$	0.00	0.983
Π_3	$r = 0$	77.39	0.000
	$r \leq 1$	33.24	0.000
	$r \leq 2$	0.01	0.899

Simulation: Π_1 model

Fitted model Π_1 with unrestricted α, β:

Para-meter	True value	Unrestricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.471	0.072	< 0.001
α_2	0	0.074	0.075	0.329
α_3	0	-0.121	0.077	0.117
β_1	1	1		
β_2	-1	-1.028		
β_3	0	0.031		
μ_1	6	6.321	0.214	< 0.001
μ_2	5	4.810	0.224	< 0.001
μ_3	5	5.209	0.230	< 0.001

$$\Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\Pi}_1 = \begin{pmatrix} -0.471 & 0.484 & -0.015 \\ 0.074 & -0.076 & 0.002 \\ -0.121 & 0.124 & -0.004 \end{pmatrix}$$

Fitted model Π_1 with restricted α, β:

Para-meter	True value	Restricted α, β Estimate	Std. Error	p value
α_1	-0.5	-0.469	0.072	< 0.001
α_2	0	0		
α_3	0	0		
β_1	1	1		
β_2	-1	-1		
β_3	0	0		
μ_1	6	6.066	0.180	< 0.001
μ_2	5	5.006	0.188	< 0.001
μ_3	5	4.886	0.193	< 0.001

$$\Pi_1 = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{\Pi}_1 = \begin{pmatrix} -0.469 & 0.469 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Conclusion: We recover the correct uni-directional coupling structure.

Standard measures of coupling

The *mean phase coherence measure* is

$$R(\phi_{it}, \phi_{jt}) = \left| \frac{1}{N} \sum_{n=1}^{N} e^{i(\phi_{i,t_n} - \phi_{j,t_n})} \right|$$

Note: symmetrical, no direction.

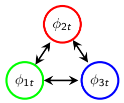
Question: Does the co-integration analysis provide more power to detect coupling?

We look at model

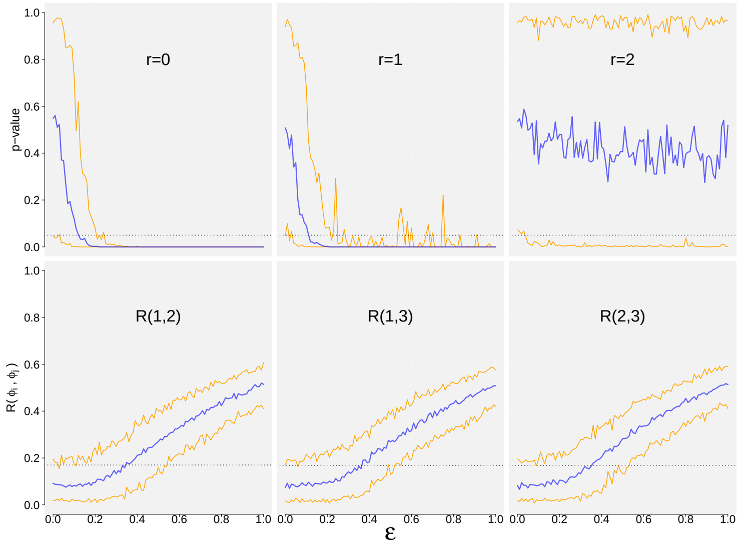
$$\Pi = \varepsilon \Pi_3, \quad \varepsilon \in [0,1]$$

where

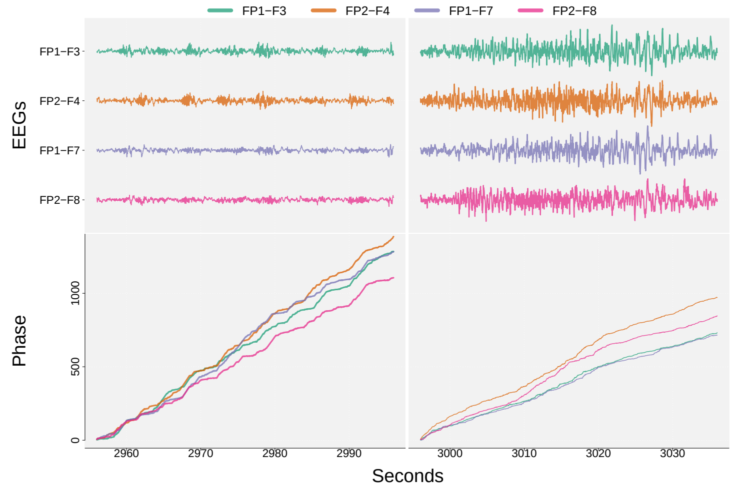
Π_3 :



Detecting coupling



Analysis of EEG data



Analysis of EEG data

	Prior to seizure		During seizure	
H_r	Test values	p-value	Test values	p-value
$r = 0$	105.87	0.000	1132.64	0.000
$r \leq 1$	42.82	0.000	41.68	0.008
$\mathbf{r \leq 2}$	**9.98**	**0.053**	**7.19**	**0.618**
$r \leq 3$	0.46	0.439	0.72	0.786

Tabel: Rank tests for EEG phases in the bottom of previous Figure. The rank is determined to $r = 2$ in both periods, although the conclusion is far stronger during the seizure. The significance of the statistics are found using 5000 bootstrap samples prior to the seizure due the border limit case of around 5%, during the seizure the p-value is determined from 2000 bootstrap samples.

Analysis of EEG data

Para-meter	Prior to seizure Estimate	Std. Error	p value	During seizure Estimate	Std. Error	p value
$\alpha_{\text{FP1-F3},1}$	-0.100	0.018	< 0.001	-0.462	0.028	< 0.001
$\alpha_{\text{FP1-F7},1}$	-0.002	0.019	0.930	-0.308	0.032	< 0.001
$\alpha_{\text{FP2-F4},1}$	-0.035	0.017	0.044	-0.722	0.035	< 0.001
$\alpha_{\text{FP2-F8},1}$	-0.115	0.030	< 0.001	-0.648	0.042	< 0.001
$\alpha_{\text{FP1-F3},2}$	-0.117	0.016	< 0.001	0.041	0.033	0.212
$\alpha_{\text{FP1-F7},2}$	-0.024	0.016	0.147	0.071	0.037	0.057
$\alpha_{\text{FP2-F4},2}$	-0.026	0.015	0.084	0.173	0.041	< 0.001
$\alpha_{\text{FP2-F8},2}$	-0.049	0.026	0.063	0.468	0.049	< 0.001
$\beta_{\text{FP2-F4},1}$	-3.424			-0.036		
$\beta_{\text{FP2-F8},1}$	2.610			-0.573		
$\beta_{\text{FP2-F4},2}$	2.486			-0.840		
$\beta_{\text{FP2-F8},2}$	-3.631			0.188		
$\mu_{\text{FP1-F3}}$	25.210	2.162	< 0.001	39.647	1.307	< 0.001
$\mu_{\text{FP1-F7}}$	30.648	2.252	< 0.001	36.499	1.473	< 0.001
$\mu_{\text{FP2-F4}}$	39.058	2.107	< 0.001	58.268	1.608	< 0.001
$\mu_{\text{FP2-F8}}$	48.853	3.615	< 0.001	54.765	1.947	< 0.001

Outlook: Challenges

- Interpret cointegration models for coupled oscillators.
- Derive a non-linear cointegration mechanism to model Kuramoto.
- Derive a framework with non-linear deterministic trends for the model.
- Extend to high-dimensional systems.

Reference:

Jacob Østergaard, Anders Rahbek and Susanne Ditlevsen (2017): Oscillating systems with cointegrated phase processes. Journal of Mathematical Biology.